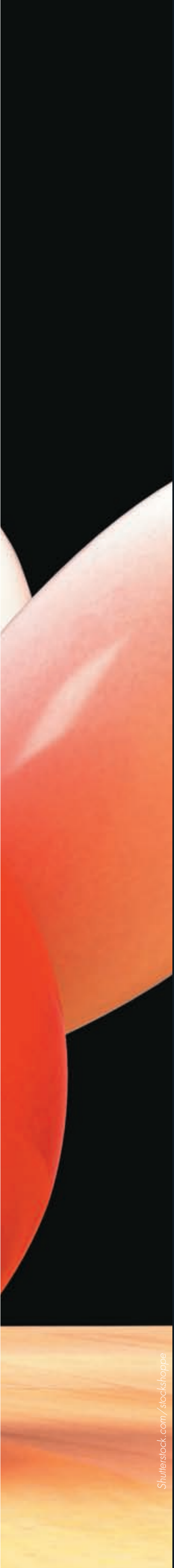
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8

RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

TERMINOLOGY

acceleration
angular velocity
differential equation
displacement
first-order
force
frequency
general solution
gradient field
implicit differentiation
logistic equation
logistic model
mass
momentum
Newton's laws of motion
position
related variables
separable variables
simple harmonic motion
slope field
velocity

Prior learning

RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

- use implicit differentiation to determine the gradient of curves whose equations are given in implicit form (ACMSM128)
- related rates as instances of the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (ACMSM129)
- solve simple first order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and, in general, differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables (ACMSM130)
- examine slope (direction or gradient) fields of a first order differential equation (ACMSM131)
- formulate differential equations including the logistic equation that will arise in, for example, chemistry, biology and economics, in situations where rates are involved. (ACMSM132)

MODELLING MOTION

- examine momentum, force, resultant force, action and reaction (ACMSM133)
- consider constant and non-constant force (ACMSM134)
- understand motion of a body under concurrent forces (ACMSM135)
- consider and solve problems involving motion in a straight line with both constant and non-constant acceleration including simple harmonic motion and the use of expressions $\frac{dv}{dt}$, $v\frac{dv}{dx}$ and $\frac{d\left(\frac{1}{2}v^2\right)}{dx}$ for acceleration. (ACMSM136) AC

8.01 IMPLICIT DIFFERENTIATION

You cannot differentiate the equation of a circle such as $x^2 + y^2 = 25$ in the usual way because it is not in the form $y = f(x)$. You can still find $\frac{dy}{dx}$ by differentiating the whole equation and using the chain rule for the parts that are functions of y.

For the equation $x^2 + y^2 = 25$, you get $2x + 2y \times \frac{dy}{dx} = 0$, which gives $\frac{dy}{dx} = -\frac{x}{y}$. This has the disadvantage of needing both the coordinates of a point on the line to calculate the derivative.

IMPORTANT

Implicit differentiation is the technique of differentiating an equation, involving the dependent and independent variables, using the chain rule for the parts that are functions of the dependent variable.

Example 1

Find $\frac{dy}{dx}$ for $x^2 + 2xy - y^2 = 4$.

Solution

Differentiate each term with respect to x.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

Use the product rule for the second term.

$$2x + \left[\frac{d}{dx}(2x) \times y + 2x \times \frac{d}{dx}(y)\right] - 2y\frac{dy}{dx} = 0$$

Simplify.

$$2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

Now solve for $\frac{dy}{dx}$.

$$2x\frac{dy}{dx} - 2y\frac{dy}{dx} = -2x - 2y$$

Factorise the LHS.

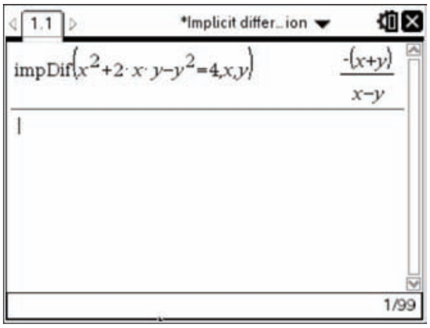
$$\frac{dy}{dx}(2x - 2y) = -2x - 2y$$

Make $\frac{dy}{dx}$ the subject and simplify.

$$\frac{dy}{dx} = \frac{-2x - 2y}{2x - 2y} = \frac{-x - y}{x - y} = \frac{x + y}{y - x}$$

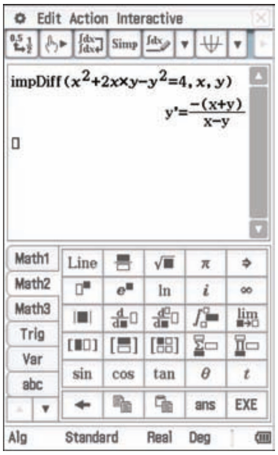
TI-Nspire CAS

Use menu, 4: Calculus, E: Implicit Differentiation and complete with the equation, independent and dependent variables. Make sure that you avoid the variable xy by including the $\times$ between x and y.



ClassPad

Tap **Action**, **Calculation** and **impDiff** and enter in turn the equation, independent and dependent variables, with each entry separated by a comma. Close the parentheses and press EXE. Make sure that you avoid the variable xy by including a $\times$ between x and y.



You don't usually write the equations of circles, ellipses and hyperbolas in the form $y = f(x)$, so you use implicit differentiation for these relations. This is particularly useful in Physics for tangents and normals in applications such as finding the reflection from curved surfaces. Remember that the normal is perpendicular to the tangent and the product of their gradients is -1.

Example 2

a Find the gradient of the tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point $(4, \frac{9}{5})$.

b Find the equation of the normal to $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point $(4, \frac{9}{5})$.

Solution

a Use implicit differentiation. $\quad \frac{2x}{25} + \frac{2y}{9}\frac{dy}{dx} = 0$

Make $\frac{dy}{dx}$ the subject. $\quad \frac{dy}{dx} = \frac{-9x}{25y}$

Find the gradient of the tangent. $\quad m_t = \frac{dy}{dx} = \frac{-9 \times 4}{25 \times \frac{9}{5}} = -\frac{4}{5}$

b Use $m_t m_n = -1$ to find the slope of the normal. $\quad m_n = \frac{5}{4}$

Write the slope–point form of a straight line. $\quad y - y_1 = m(x - x_1)$

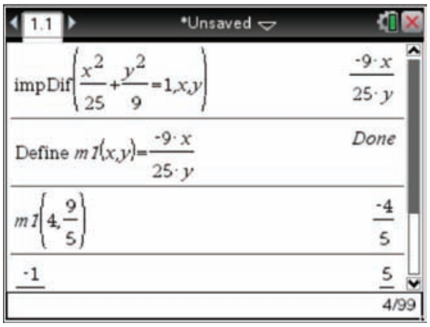
Substitute m_n and $(4, \frac{9}{5})$. $\quad y - \frac{9}{5} = \frac{5}{4}(x - 4)$

Simplify. $\quad y = \frac{5}{4}x - \frac{16}{5}$

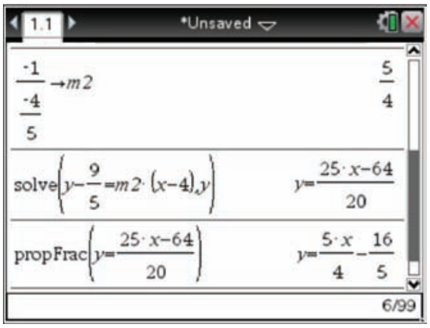
Write in standard form. $\quad 25x - 20y - 64 = 0$

TI-Nspire CAS

a Use implicit differentiation to find the derivative. Define $m1(x,y)$ to be the derivative.
Find the value at $(4, \frac{9}{5})$.



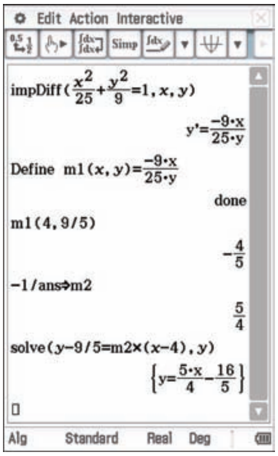
b Use −1/ans to find the gradient of the normal. Solve the two-point form of the equation of a straight line for y and use menu, 2: Number, 7: Fraction Tools and 1: Proper Fraction to change to the slope–intercept form.



ClassPad

a Use implicit differentiation to find the derivative. Define m1(x,y) to be the derivative using copy and paste. Find the value of m1 at $(4, \frac{9}{5})$.

b Use −1/ans to find the gradient of the normal and assign this to the name m2. Solve the two-point form of the equation of a straight line for y with its gradient given by m2.



You can also find the equation of the tangent by using a CAS calculator to draw the relation as a conic/ellipse. You draw the tangent at the desired point to get its equation.

EXERCISE 8.01 Implicit differentiation

Concepts and techniques

WS Implicit differentiation

1 Example 1 Find the derivative of each of the following.

a $3x^2 - 5xy + y^3 = 3$

b $x^2y - yx^2 = xy$

c $\frac{2}{x} + \frac{3}{y} = 5$

d $2x\sqrt{y} = (1 - y)^2$

2 Consider the function $x^2 + 2y = 5$.

a Differentiate the function implicitly to find $\frac{dy}{dx}$.

b Find y explicitly in terms of x and show that $\frac{dy}{dx}$ is the same as that in part **a**.

3 Consider the relation $y^2 + 2y - x = 3$.

a Differentiate the relation implicitly to find $\frac{dy}{dx}$.

b i Find x in terms of y.

ii Hence find $\frac{dy}{dx}$.

4 Find $\frac{dy}{dx}$ in terms of x and y for each of the following by implicit differentiation.

a $\sin(x + y) = \tan(y)$

b $e^{2y}x - 2xy = 3$

c $\log_e(x + y) = 5x$

d $x^2\cos(y) = e^y$

5 Example 2 a Find the gradient of the tangent to the hyperbola with equation $x^2 - y^2 = 16$ at the point $(5, -3)$.

b Hence find the equation of the normal to $x^2 - y^2 = 16$ at the point $(5, -3)$.

6 Find the equation of the tangent to the curve with equation $\frac{(x+1)^2}{36} - \frac{(y-2)^2}{16} = 1$ at $\left(\frac{13}{2}, 5\right)$.

7 Find the equations of the tangents to the ellipse with equation $4x^2 + 9y^2 = 40$ that have a gradient of $-\frac{2}{9}$.

8 Find the equation of the normal to the curve with equation $\frac{(x-3)^2}{25} + \frac{(y+2)^2}{16} = 1$ at $\left(6, \frac{6}{5}\right)$.

9 Find the equations of the tangents to the ellipse with equation $\frac{9x^2}{52} + \frac{16y^2}{52} = 1$ that are parallel to the line $y = \frac{9}{8}x - \frac{1}{8}$.

10 Find the intersection of the normals to the curve with equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $y = 4$.

Reasoning and communication

11 For the curve with equation $y^2 = 4ax$, show that the equation of the tangent at a point $P(x_1, y_1)$ on the parabola is $y_1y = 2a(x + x_1)$.

12 Consider the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the equation of the tangent to the ellipse with slope m is $y = mx \pm \sqrt{a^2m^2 + b^2}$.

13 Show that the equation of the tangent, with gradient m, to the curve with equation $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.

14 Show that the equation of the tangent to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point $P(x_1, y_1)$ on the ellipse is $b^2x_1x + a^2y_1y - a^2b^2 = 0$.

8.02 RELATED RATES OF CHANGE

You will sometimes need to find the rate of change of one variable with respect to time from the rate of change of a related variable with respect to time.

IMPORTANT

The relation between rates of change of **related variables** with respect to time can be found using the chain rule in the form $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

Example 3

Given $y = 2x^2 - 3x + 1$ and $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = -2$.

Solution

Find the rate of change of y with respect to (wrt) x.

$$\frac{dy}{dx} = 4x - 3$$

Express the rate of change of y wrt t in terms of the rate of change wrt to x.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Substitute the derivatives.

$$= 5(4x - 3)$$

Substitute in the value of x.

$$= 5[4 \times (-2) - 3]$$

Write the answer.

$$= -55$$

Example 4

A spherical metal ball is heated so that its radius is expanding at the rate of 0.04 mm per second. At what rate will its volume be increasing when the radius is 3 mm?

Solution

Write the volume of the sphere.

$$V = \frac{4}{3}\pi r^3$$

Find the rate of change with wrt to radius.

$$\frac{dV}{dr} = 4\pi r^2$$

Express the rate of change of (wrt) time in terms of the rate of change wrt to radius.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Substitute the values, including the radius.

$$= 4\pi \times 3^2 \times 0.04$$

Calculate the answer.

$$= 1.44\pi \text{ mm}^3/\text{s}$$

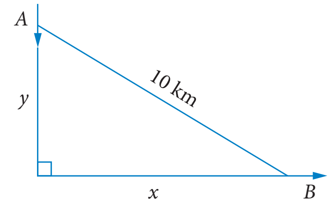
Example 5

Car A is north of an intersection, while car B is moving east at 60 km/h, away from the intersection. The distance between the cars remains constant at 10 km. Find the speed of car A when car B is 8 km from the intersection.

Shutterstock.com/Tim Roberts Photography

Solution

Draw a diagram to represent the situation.



Write the relationship between x and y.

$$x^2 + y^2 = 10^2$$

Differentiate with respect to time.

$$2x \times \frac{dx}{dt} + 2y \times \frac{dy}{dt} = 0$$

Rearrange.

$$\frac{dy}{dt} = \frac{dx}{dt} \times \left(-\frac{x}{y}\right)$$

Find y when $x = 8$.

$$8^2 + y^2 = 100 \text{ so } y = 6$$

Substitute values into the rates.

$$\frac{dy}{dt} = 60 \times \left(-\frac{8}{6}\right) \text{ km/h}$$

Calculate the answer.

$$= -80 \text{ km/h}$$

Comment on the minus sign.

The negative means Car A is travelling towards the intersection.

Write the answer.

Car A is travelling at 80 km/h towards the intersection.

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EXERCISE 8.02 Related rates of change

Concepts and techniques

WS Related rates

1 Example 3 Find an expression for $\frac{dy}{dt}$ given

a $y = x^4$ and $\frac{dx}{dt} = 2$

b $y = e^{2x}$ and $\frac{dx}{dt} = 5$

2 Evaluate $\frac{dy}{dt}$ at $x = 4$, given

a $y = 2x^3 + 3x - 7$ and $\frac{dx}{dt} = 3$

b $y = (3x + 1)^3$ and $\frac{dx}{dt} = -4$

c $y = x\log_e(x)$ and $\frac{dx}{dt} = 3$

3 If $y = \sin(2x)$ and $\frac{dx}{dt} = 3$, evaluate $\frac{dy}{dt}$ when $x = \frac{\pi}{8}$.

Reasoning and communication

4 Example 4 A cube is expanding so that its side is increasing at the constant rate of 0.12 mm/s. Find the rate of increase in its volume when its side is 150 mm.

5 The radius of a cylindrical pipe 2 m long expands with heat at a constant rate of 1.2×10^{-3} mm/s. Find the rate at which the volume of the pipe will be increasing when its radius is 19 mm.

6 Find the rate of change of the surface area of a balloon when its radius is 6.3 cm, if its radius is expanding at a constant rate of 1.3 cm s^{-1}.

7 A cone's volume is given by $V = \frac{6\pi h^2}{7}$, where h is the height of liquid in the cone. If the height of the liquid is increasing at a rate of 2.3 cm/s, find the rate of increase in the volume of the liquid when its height is 12.9 cm.

8 A point, P, moves along a curve with equation $y = 2x^2 - 7x + 9$. What will the rate of change in the y-coordinate of P be when the x-coordinate is increasing at a rate of 8 units per second and the value of x is 3?

9 Example 5 A 4 m long ladder is leaning against a wall. Its base is slipping away from the wall at a constant rate of 2 m/s. At what rate, correct to 2 decimal places, will the top of the ladder be slipping down the wall when the base is 1 m out from the wall?

Alamy/artpartner-images.com

10 A particle is moving so that its velocity is given by the formula $v = 8x^3 - 5x^2 - 3x - 1$, where x is its displacement. If the rate of change of the displacement is a constant 4.2 cm/s, find the rate of change in velocity when the displacement is −4.7 cm.

11 A population increases at a constant rate of 15 000 people per year. If the population has the formula $P = x^2 - 3000x + 100$, where x is the number of houses available, find the rate at which the number of houses will be increasing when there are 5000 houses.

12 The area of an equilateral triangle is increasing at the rate of $2 \text{ cm}^2 \text{ s}^{-1}$.

a Show that the area of the triangle is given by $A = \dfrac{\sqrt{3}x^2}{4}$, where x is the side of the triangle.

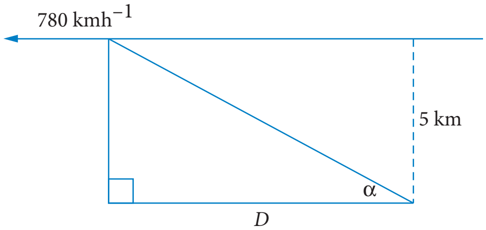
b Find the rate of increase of the side of the triangle when it has side length $2\sqrt{3}$ cm.

13 The surface area of a spherical bubble is increasing at a constant rate of $1.9 \text{ mm}^2 \text{ s}^{-1}$. Find the rate of increase in its volume when its radius is 0.6 mm.

14 A spherical bath oil capsule dissolves in the bath so that its decrease in volume is proportional to its surface area. If its shape remains spherical as it dissolves, show that the radius of the capsule will decrease at a constant rate.

15 Water is leaking out of a right circular cone at a constant rate of $2 \text{ cm}^3\text{s}^{-1}$. The cone has a radius of 10 cm and height of 30 cm. Find the rate at which the radius is decreasing when the height is 3 cm.

16 Justin sees a plane fly directly overhead at an altitude of 5 km. The plane is moving horizontally away from Justin at a constant speed of 780 km h^{-1} with an angle of elevation of α as shown.



a Show that the horizontal distance D between the plane and Justin is given by $D = \dfrac{5}{\tan(\alpha)}$.

b Find the simplest expression for $\dfrac{dD}{d\alpha}$.

c Find the rate at which the angle of elevation is changing over time (in radians/hour) when $\alpha = \dfrac{\pi}{6}$.

d Is this a reliable measure of the rate in the long term?

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8.03 DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)$

You have already solved some simple **differential equations** by finding indefinite integrals (antiderivatives). You can already solve the differential equation $\frac{dy}{dx} = 6x^2 - 2x + 9$.

IMPORTANT

A **differential equation** involves one or more derivatives of a function. A **first-order** differential equation involves only the first derivative. The general solution of a differential equation is the function itself.

The general solution of the equation $\frac{dy}{dx} = 6x^2 - 2x + 9$ is $y = 2x^3 - x^2 + 9x + c$. A specific solution can only be found if you have other information that you can use to find the value of the constant of integration, c.

Example 6

Find the general solutions of the following differential equations.

a $\frac{dy}{dx} = (1-2x)^3$ **b** $\frac{dy}{dt} = 6e^{-3t} + 4e^{2t}$ **c** $\frac{dp}{dy} = \frac{1}{\sqrt{9-y^2}}$

Solution

a Find the antiderivative. $\int (1-2x)^3 dx = \frac{(1-2x)^4}{4\times(-2)} + c$

Simplify to write the answer. $y = -\frac{1}{8}(1-2x)^4 + c$

b Find the indefinite integral. $\int 6e^{-3t} + 4e^{2t} dt = -2e^{-3t} + 2e^{2t} + c$

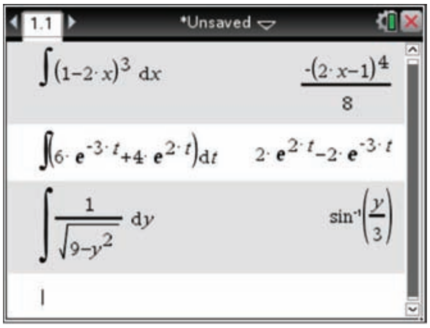
Write in the usual order. $y = 2e^{2t} - 2e^{-3t} + c$

c Use $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$. $\int \frac{1}{\sqrt{3^2 - y^2}} dy = \sin^{-1}\left(\frac{y}{3}\right) + c$

Write the answer. $p = \sin^{-1}\left(\frac{y}{3}\right) + c$

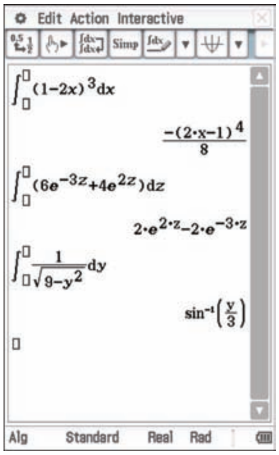
TI-Nspire CAS

Use menu, 4: Calculus, 3: Integral, but do not fill in the limits. The calculator does not put the '+ c' on the integral.
Make sure that the calculator is set in radians for part **c**.



ClassPad

Tap **Interactive**, **Calculation**, then $\int$ and choose indefinite integral, or use [$\int$ key] from the [Math2] soft keyboard but do not fill in the limits. The calculator does not put '+ c' on the integral. As t is a reserved variable, use z for part **b**. Make sure that the calculator is set to **Rad** mode for part **c**.



To find a particular solution to a differential equation, a point on the curve is required. To find the value of c when $y=\int\left(6x^2-2x+9\right)dx=2x^3-x^2+9x+c$ and (0, 6) is on the line (that is, $y(0)=6$), substitute $x=0$ and $y=6$ into $y=2x^3-x^2+9x+c$, giving $c=6$ so $y=2x^3-x^2+9x+6$.

A general solution to a differential equation generates the equations for a whole family of curves with different constants c, and a particular solution gives the equation for one of those curves with a particular value of c.

Example 7

Find the solution to the following differential equations.

a $\dfrac{dy}{dx}=\dfrac{x-2}{x^2-4x}$ if $y(2)=1$

b $\dfrac{dy}{dx}=\dfrac{1}{x^2+5x+6}$ if $y(1)=0$

c $\dfrac{dy}{dx}=\cos^2(x)$ if $y\left(\frac{\pi}{6}\right)=0$

Solution

a Use $\int\left(\dfrac{f'(x)}{f(x)}\right)dx=\log_e|f(x)|+c$ $\qquad$ $\int\left(\dfrac{x-2}{x^2-4x}\right)dx=\frac{1}{2}\int\left(\dfrac{2x-4}{x^2-4x}\right)dx$

Simplify. $\qquad$ $y=\frac{1}{2}\log_e|x^2-4x|+c$

Use the value to find c. $\qquad$ $1=\frac{1}{2}\log_e|2^2-4\times 2|+c$

Solve for c. $\qquad$ $c=1-\log_e(2)$

Write the answer. $\qquad$ $y=\frac{1}{2}\log_e|x^2-4x|+1-\log_e(2)$

b	Factorise $x^2 + 5x + 6$.	$x^2 + 5x + 6 = (x+3)(x+2)$
	Change $\dfrac{1}{x^2+5x+6}$ to partial fractions.	$\dfrac{1}{x^2+5x+6} = \dfrac{A}{x+3} + \dfrac{B}{x+2}$ $= \dfrac{A(x+2)+B(x+3)}{(x+2)(x+3)}$
	Solve for A and B.	$A(x+2) + B(x+3) = 1$
	Substitute $x = -3$ and $x = -2$ to find A and B.	$A = -1$ and $B = 1$
	Write the integral and use the expression with partial fractions.	$\int \dfrac{1}{x^2+5x+6}dx = \int \left[\dfrac{1}{x+2} - \dfrac{1}{x+3}\right]dx$
	Separate the terms.	$= \int \dfrac{1}{x+2}dx - \int \dfrac{1}{x+3}dx$
	Use $\int \dfrac{1}{x}dx = \log_e \lvert x \rvert + c$.	$= \log_e \lvert x+2 \rvert - \log_e \lvert x+3 \rvert + c$
	Substitute (1, 0).	$0 = \log_e(3) - \log_e(4) + c$
	Solve for c.	$c = \log_e(4) - \log_e(3)$ $= \log_e\left(\dfrac{4}{3}\right)$
	Write the expression.	$\int \dfrac{1}{x^2+5x+6}dx = \log_e \lvert x+2 \rvert - \log_e \lvert x+3 \rvert + \log_e\left(\dfrac{4}{3}\right)$
	Simplify using log laws.	$= \log_e \left\lvert \dfrac{4(x+2)}{3(x+3)} \right\rvert$
	Write the answer.	$y = \log_e \left\lvert \dfrac{4(x+2)}{3(x+3)} \right\rvert$
c	Write the integral and use the formula $\cos(2x) = 2\cos^2(x) - 1$.	$\int \cos^2(x)dx = \dfrac{1}{2}\int [\cos(2x)+1]dx$
	Simplify.	$= \dfrac{1}{4}\sin(2x) + \dfrac{1}{2}x + c$
	Substitute $(\frac{\pi}{6}, 0)$ to find c.	$0 = \dfrac{1}{4}\sin(2 \times \dfrac{\pi}{6}) + \dfrac{1}{2} \times \dfrac{\pi}{6} + c$
	Solve for c.	$c = -\dfrac{\sqrt{3}}{8} - \dfrac{\pi}{12}$
	Write the answer.	$y = \dfrac{1}{4}\sin(2x) + \dfrac{1}{2}x - \dfrac{\sqrt{3}}{8} - \dfrac{\pi}{12}$

Example 8

A rocket is constructed so that its acceleration for a short time after take-off is given by $a = 10t + 2$ ms^{-2}, where t is in seconds. It takes off vertically from a height of 100 m above sea level. Find its velocity and height after 4 s.

Solution

Write the integral. $v = \int (10t + 2)\,dt$

Simplify. $= 5t^2 + 2t + c_1$

Substitute $v(0) = 0$ to find c_1. $c_1 = 0$

Write the equation. $v = 5t^2 + 2t$

Substitute $t = 4$ into the equation. $v = 88$ m/s

The velocity is the derivative of the height. $\frac{dh}{dt} = 5t^2 + 2t$

Find the integral. $h = \frac{5}{3}t^3 + t^2 + c_2$

Substitute $h = 100$ at $t = 0$ to find c_2. $c_2 = 100$

Write the equation. $h = \frac{5}{3}t^3 + t^2 + 100$

Find the value at $t = 4$. $h = \frac{5}{3} \times 64 + 16 + 100 = 222\frac{2}{3}$

Write the answer. The velocity and height after 4 s are 88 m/s and $222\frac{2}{3}$ m above sea level respectively.

EXERCISE 8.03 Differential equations of the form $\frac{dy}{dx} = f(x)$

Concepts and techniques

WS Simple differential equations

1 Example 6 Find the general solutions to the following differential equations.

a $\frac{dy}{dx} = 3x^2 - 4x + 7$

b $\frac{dy}{dt} = 8\cos(3t)$

c $\frac{dy}{dw} = 24w^3 - \frac{12}{w}$

d $\frac{dy}{dx} = 6e^{-3x} - 8e^{2x}$

e $\frac{dy}{dx} = (5 + 2x)^4$

f $\frac{dy}{dx} = \frac{1}{(3-2x)^3}$

g $\frac{dx}{dy} = \frac{4}{16+y^2}$

h $\frac{dx}{dy} = \frac{-1}{\sqrt{4-y^2}}$

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2 Example 7 Find the solutions to the following differential equations.

a $\frac{dy}{dx}=\frac{4x-10}{x^2-5x}$ if $y(6) = 0$

b $\frac{dy}{dx}=\frac{3x+1}{x^2}$ if $y(1) = 3$

c $\frac{dy}{dx}=\frac{6}{(x-2)(x+3)}$ if $y(3) = 1$

d $\frac{dy}{dx}=\frac{3}{x^2+4x+6}$ if $y(0) = 2$

e $\frac{dy}{dx}=\tan^2(x)$ if $y\left(\frac{\pi}{4}\right)=1$

f $\frac{dy}{dx}=8\cos(3x)\sin(3x)$ if $y\left(\frac{\pi}{2}\right)=2$

g $\frac{dx}{dy}=\frac{1}{\sqrt{4-9y^2}}$ if when $y=0$, $x=1$

h $\frac{dx}{dy}=\frac{3}{25+4y^2}$ if when $y=0$, $x=1$

i $\frac{dy}{dx}=10x\left(5x^2-2\right)^4$ if $y(1)=48$

j $\frac{dy}{dx}=-3\cos^2(x)\sin(x)$ if $y\left(\frac{\pi}{3}\right)=0$

Reasoning and communication

3 a Express $\frac{x^3+1}{x^2-1}$ as a partial fraction.

b Hence find the general solution to $\frac{dy}{dx}=\frac{-x^3-1}{x^2-1}$.

4 a Find the derivative of $x\log_e(x)$.

b Hence find the general solution to $\frac{dy}{dx}=\log_e(x)$.

5 Example 8 The velocity of a stone thrown up into the air is given by the equation $v = 20 - 10t$ m s^{-1}, where t is in seconds. The stone is thrown from a height of 1.5 m. Use a differential equation to find an expression for the height of the stone, and hence find its height after 2.5 s.

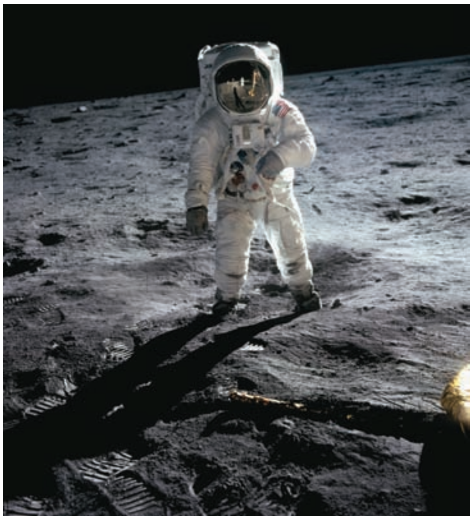
6 The marginal profit of a maker of DVD players is given by $P(n)=\frac{200n+4000}{n^2+40n+1}$, where n is the number of players made and $P(n)$ is the profit in dollars on the nth player. Use the derivative to find the total profit when n players are made, and hence the profit from 5000 players.

7 The acceleration due to gravity on the surface of the moon is such that the speed of some green cheese thrown upward by an astronaut with a strong arm is given by $v = 25 - 1.6t$, where v is in m s^{-1} and t is in seconds. Assume that the initial height is 0 m.

a Use the derivative to find the height after t s.

b Find the height after 4 s.

c Use a quadratic equation to show that the cheese never reaches a height of 200 m.



Corbis

8 The rate of storage of energy by a spring is given by $\frac{dQ}{dx} = 200x$ J m^{-1}, where Q J (joules) is the energy stored and x m is the compression of the spring. No energy is stored when there is no compression. Find the energy stored with a compression of x m, and thus the energy stored with a compression of 12 cm.

9 The marginal production cost of items is the derivative of the production cost. The marginal production cost of large tents is given by $\frac{dC}{dn} = 0.002n^2 - n + 250$, where C is the cost in dollars of producing n tents. It also costs \$20 000 to set up the production line before any tents are produced. Find the cost of producing n tents.

10 The rate of change of momentum for an object is given by the force acting on the object. An object with a momentum of 50 N s is acted upon by a force given by $F = 8 - 2t$ N for a period of 4 s. Find the momentum at t s $(0 \le t \le 4)$.

8.04 DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = g(y)$

At first glance, you may not see how to solve the differential equation $\frac{dy}{dx} = 0.05y$.

Change it to the form $\frac{1}{y}\frac{dy}{dx} = 0.05$ and integrate both sides with respect to x.

This gives $\int \frac{1}{y}\frac{dy}{dx}dx = \int 0.05dx$.

Using the rule for integration by substitution gives $\int \frac{1}{y}dy = \int 0.05dx$.

This becomes $\log_e |y| = 0.05x + c$, and assuming $y > 0$, this gives $y = e^{0.05x + c}$.

This is a very important type of differential equation with many applications. It is usually rewritten as $y = e^{0.05x+c} = e^{0.05x} \times e^c = y_0 e^{0.05x}$, where $y_0 = e^c$ is the value of y at $x = 0$.

You should also be able to see that this method applies to any equation of the type $\frac{dy}{dx} = g(y)$.

IMPORTANT

The differential equation $\frac{dy}{dx} = ky$ has the **general solution** $y = y_0 e^{kx}$, where y_0 is the value of y at $x = 0$.

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Example 9

Solve the differential equation $\frac{dy}{dx} = 5y$.

Solution

Write the equation with y on the left. $\qquad \frac{1}{y}\frac{dy}{dx} = 5$

Integrate both sides with respect to x. $\qquad \int \frac{1}{y}\frac{dy}{dx}\,dx = \int 5\,dx$

Simplify. $\qquad \log_e(y) = 5x + c$

Change to exponential form. $\qquad y = e^{5x+c}$

$\qquad = e^c \times e^{5x}$

Let $e^c = A$ and write the answer. $\qquad = Ae^{5x}$

In many real situations, the independent variable is time, so the value of A in an expression of the type $y = Ae^{kt}$ is the initial value of y.

Example 10

The rate of decay of radioactive material depends on the amount of material. The half-life is the time taken for half the material present at any time to decay. The carbon-14 half-life is 5730 years. In a charcoal sample taken from a midden, 0.05 g of carbon-14 was present. The archaeologist working on the site estimated that 0.27 g of carbon-14 was present at the time of the fire. Use a differential equation to find the age of the midden to the nearest year.

Corbis/Imaginechina

Solution

Choose the variables.	Let M = mass of material in g and t = time in years.
Decreasing gives a negative rate of change.	$\frac{dM}{dt} = -kM$
Put M on the left.	$\frac{1}{M}\frac{dM}{dt} = -k$
Integrate both sides with respect to time.	$\int \frac{1}{M}\frac{dM}{dt}dt = \int -kdt$
Simplify.	$\log_e(M) = -kt + c$
Change to exponential form.	$M = e^{-kt+c} = M_0e^{-kt}$
Use the initial value of M.	$M_0 = 0.27$
Use the half-life.	At $t = 5730$, $M = \frac{1}{2}M_0$
Substitute values.	$\frac{1}{2}M_0 = M_0e^{-5730k}$
Simplify.	$e^{-5730k} = \frac{1}{2}$
Change to logarithm form.	$-5730k = \log_e(\frac{1}{2})$ $= -\log_e(2)$
Solve for k.	$k = \frac{\log_e(2)}{5730}$
Write the model.	$M = 0.27e^{-kt}$, where $k = \frac{\log_e(2)}{5730}$
Substitute the current value of M.	$0.05 = 0.27e^{-kt}$
Simplify.	$e^{-kt} = \frac{0.05}{0.27}$
Express as a logarithm.	$-kt = \log_e\left(\frac{0.05}{0.27}\right)$
Rearrange to find t.	$t = -\log_e\left(\frac{0.05}{0.27}\right) \div \frac{\log_e(2)}{5730}$ $= 13940.857\ldots$
Write the answer.	The age, correct to the nearest year, is 13 941 years.

Functions where the rate of change is a linear function of the dependent variable also arise frequently in practical applications.

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Example 11

A piece of very hot iron was dropped into some cold water at 15°C. After 10 min, the iron had cooled to 45°C, and after a further 3 min it was at 30°C. The rate of cooling is proportional to the temperature difference between the iron and the water. Find the initial temperature of the iron in °C, correct to one decimal place.

Solution

Choose the variables.	Let temperature = T °C and time = t min
Write the rate of change equation for T.	$\frac{dT}{dt} = -k(T-15)$
Rearrange.	$\frac{1}{T-15}\frac{dT}{dt} = -k$
Integrate both sides with respect to t.	$\int \frac{1}{T-15}\frac{dT}{dt}dt = \int -k\,dt$
Use integration by substitution.	$\int \frac{1}{T-15}dT = \int -k\,dt$
Simplify.	$\log_e \lvert T-15 \rvert = -kt + c$
Change to exponential form.	$T - 15 = e^{-kt+c}$
Express as a function of t.	$T = 15 + Ae^{-kt}$
Substitute in the known values.	$45 = 15 + Ae^{-10k}$ and $30 = 15 + Ae^{-13k}$
Simplify.	$Ae^{-10k} = 30$ and $Ae^{-13k} = 15$
Divide to isolate k.	$\frac{Ae^{-10k}}{Ae^{-13k}} = \frac{30}{15}$
Simplify.	$e^{3k} = 2$
Change to logarithmic form.	$3k = \log_e(2)$, so $k = \frac{1}{3}\log_e(2)$
Substitute into the equation for $t = 10$.	$Ae^{-\frac{1}{3}\log_e(2)\times 10} = 30$
Simplify.	$A\left[e^{\log_e(2)}\right]^{-\frac{10}{3}} = 30$
Use $e^{\log_e(2)} = 2$.	$A \times 2^{-\frac{10}{3}} = 30$
Find A	$A = 30 \times 2^{\frac{10}{3}}$
Write the model.	$T = 15 + 30 \times 2^{\frac{10}{3}} \times e^{-\frac{1}{3}\log_e(2)\times t}$
Simplify.	$= 15 + 30 \times 2^{\frac{10-t}{3}}$
Substitute $t = 0$.	$T(0) = 15 + 30 \times 2^{\frac{10}{3}}$ $= 317.381\ldots$
Write the answer.	The initial temperature of the iron was 317.4 °C.

INVESTIGATION Mercury pollution

A small lake of volume 1070 ML has been polluted with approximately 200 tonnes of mercury. Water flows from a river into the lake at the rate of 20 ML h^{-1} and leaves at the same rate, helping to flush the mercury out of the lake (but into the ocean further down). The river contains no mercury and, as it enters, mixes thoroughly with the water in the lake.

- Work out a differential equation for the amount of mercury remaining in the lake, expressed as a concentration in parts per million.
- Solve the equation to find the time taken for the mercury to decrease to safe levels. You will need to consult the library to find an acceptable level of mercury pollution.
- Discuss your findings as a class group, focusing on the implications of the pollution of a largely closed environment. Consider the implications of flushing pollutants into the ocean, given that ocean currents leave the continental shelf largely undisturbed.

Alamy/David Hancock

EXERCISE 8.04 Differential equations of the form $\frac{dy}{dx} = g(y)$

WS Differential equations and exponentials

Concepts and techniques

1 Example 9 Find the general solutions of each of the following differential equations.

a $\frac{dy}{dx} = 3y$ b $\frac{dy}{dx} = 2y - 1$ c $\frac{dy}{dx} = e^{3y}$

d $\frac{dy}{dx} = y^2 + 1$ e $\frac{dy}{dx} = \sqrt{16 - y^2}$ f $\frac{dy}{dx} = y^2 + 2y$

2 Find the solution of each of the following differential equations.

a $\frac{dy}{dx} = y + 1$, if $y(0) = 1$ b $\frac{dy}{dx} = \frac{y^2 + 4y}{y + 4}$, if $y(2) = 4$

c $\frac{dy}{dx} = y^2 + 6y + 5$, if $y(1) = -6$ d $\frac{dy}{dx} = \sqrt{1 - y^2}$, if $y\left(\frac{5\pi}{6}\right) = -\frac{1}{2}$

Reasoning and communication

3 Example 10 The amount of carbon-14 remaining in a sample of bone is 20% of the original amount assumed to be present. Carbon-14 has a half-life of 5730 years and decays at a rate proportional to the quantity present. Use a differential equation to find the quantity present after time t and hence find the age of the bone.

4 A lake has become polluted with phosphates and an algal bloom begins to affect it. The algae covered only 1.5 m^2 on 3 October. They grow at a rate proportional to the quantity present. On 15 October they covered 4 m^2. Use a differential equation to find an expression for the area covered after time t. How long will it take to entirely cover the 5 km^2 lake surface, if growth continues in this fashion?

Newspix/Bob Barker

5 Bacteria grow by cell division, so the growth of a bacterial colony is proportional to the number present. In a deep wound, 20 anaerobic *Clostridium* bacteria are initially present. After 2 h, the number has grown to 90. Use a differential equation to determine the number present after time t. When the number present reaches 1000, the patient begins to develop symptoms of nerve poisoning from the toxins produced by the bacteria. How long does this take?

6 A milk sample is taken to check the level of certain bacteria. The sample was carried in the warm boot of a car for 6 h before it was checked. When checked, the level of bacteria present was found to be 200 times the acceptable amount. After a further hour, the level had risen even further to 250 times the acceptable amount. The dairy farmer from whom the sample originated claimed that, since the sample had been stored in ideal growing conditions before testing, the results were invalid. Assuming that the growth is proportional to the number of bacteria present, use a differential equation for the level at time t to find whether the sample was outside acceptable limits at the time it was taken.

7 Example 11 An oven is heated at a constant rate of 3600 J s^{-1} (watts, W) from its initial temperature of 20°C. At the same time, the oven loses heat to its surroundings at the rate of $7.5(T - 20)$ W, where T is the temperature of the oven. Each 6000 J of energy retained by the oven increases the temperature by 1°C. Use a differential equation to find the temperature of the oven after 5 minutes, and the eventual temperature.

8 The rate at which a hot object cools is approximately proportional to the difference between its temperature and the temperature of the surroundings. A cup of black coffee is made with boiling water. After 2 minutes, its temperature has dropped to 80°C. The room is at 25°C. Use a differential equation to find how much longer it will take to drop to 50°C.

9 A furnace is switched on at 9 a.m. Heat is supplied at a constant rate; but as the furnace temperature increases, heat is lost at a rate determined by the difference between its temperature and the surrounding temperature. The temperature of both was 20°C at 9 a.m. At 9:30 a.m. the furnace was at 200°C and by 10 a.m. it was at 350°C. The minimum operating temperature is 800°C. Use a differential equation to find the time when the furnace was ready for use and the highest temperature it could be expected to reach.

Alamy/Janine Wiedel Photolibrary

10 A dust particle of mass m falls according to the equation

$$m\frac{dv}{dt} = mg - kv$$

where m is the mass, g is the acceleration due to gravity, v is the velocity and k is a constant. Assuming that the particle falls from rest, solve the equation and comment on the solution.

8.05 FIRST ORDER DIFFERENTIAL EQUATIONS WITH SEPARABLE VARIABLES

You can separate the dependent and independent variables in the differential equation $\frac{dy}{dx} = (x-1)(y+3)$ by writing it as $\frac{1}{y+3}\frac{dy}{dx} = x-1$.

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IMPORTANT

A differential equation with **separable variables** can be rewritten so that the dependent and independent variables are separated. These are often of the form $\frac{dy}{dx} = f(x) \cdot g(y)$.

You can integrate both sides of $\frac{1}{y+3}\frac{dy}{dx} = x - 1$ to give $\int \frac{1}{y+3}\frac{dy}{dx}dx = \int (x-1)dx$, so $\int \frac{1}{y+3}dy = \int (x-1)dx$, which gives $\log_e(y+3) = \frac{1}{2}x^2 - x + c$ and the solution $y = Ae^{\frac{1}{2}x(x-2)} - 3$.

Strictly speaking, we should write the integral of $\frac{1}{y+3}$ as $\log_e|y+3|$. If y was less than -3, you would have to make the integral $\log_e(-y-3)$ but this would be obvious from the context.

Example 12

Solve $\frac{dy}{dx} = (2x-3)(3y+4)$.

Solution

Separate the variables.	$\frac{1}{3y+4}\frac{dy}{dx} = 2x - 3$
Integrate both sides.	$\int \frac{1}{3y+4}\frac{dy}{dx}dx = \int (2x-3)dx$
Use integration by substitution.	$\int \frac{1}{3y+4}dy = \int (2x-3)dx$
It is easiest if the coefficient of y is 1.	$\frac{1}{3}\int \frac{1}{y+\frac{4}{3}}dy = \int (2x-3)dx$
Simplify.	$\frac{1}{3}\log_e\left\|y+\frac{4}{3}\right\| = x^2 - 3x + c$
Multiply both sides by 3 to simplify.	$\log_e\left\|y+\frac{4}{3}\right\| = 3x^2 - 9x + 3c$
Assume $y + \frac{4}{3} > 0$ and change form.	$y + \frac{4}{3} = Ae^{3x(x-3)}$, where $A = e^{3c}$
Express in the usual way.	$y = Ae^{3x(x-3)} - \frac{4}{3}$

You still need some other information to find specific solutions to differential equations with separable variables.

Example 13

The rate at which a new social network expands is proportional to both the size of the network and the time since its establishment. Two weeks after the establishment of a new network, there are 30 members and after another week there are 120. How long it will take to pass the 10 000 member milestone?

Solution

Choose the variables.	Let m be the number of members and t = time in weeks
Write the rate equation for m.	$\frac{dm}{dt} = kmt$
Separate the variables.	$\frac{1}{m}\frac{dm}{dt} = kt$
Integrate both sides.	$\int \frac{1}{m}\frac{dm}{dt}dt = \int ktdt$
Simplify.	$\log_e(m) = \frac{1}{2}kt^2 + c$
Write $K = \frac{1}{2}k$ and change form.	$m = Ae^{Kt^2}$
Substitute in the known values.	$30 = Ae^{4K}$ and $120 = Ae^{9K}$
Divide to find K.	$e^{5K} = 4$
Solve for K.	$K = 0.2\log_e(4)$
Substitute in $t = 2$ to find A.	$30 = Ae^{4\times0.2\log_e(4)}$
Find A.	$A = 30 \div 4^{0.8} = 9.896\ldots$
Write the model.	$m = 9.896\ldots e^{0.2t\times\log_e(4)}$
Substitute $m = 10\,000$.	$10\,000 = 9.896\ldots e^{0.2\log_e(4)\times t}$
Solve for t.	$t = 24.952\ldots$
Write the answer.	It will take about 25 weeks to get to 10 000 members.

In some cases, you will need to use more sophisticated techniques to solve differential equations with separable variables.

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Example 14

Solve $\frac{dp}{dq} = 2pq(p+3)$ if $p = 10$ when $q = 3$.

Solution

Separate the variables.

$$\frac{1}{p(p+3)}\frac{dp}{dq} = 2q$$

Integrate both sides with respect to q.

$$\int \frac{1}{p(p+3)}\frac{dp}{dq}\,dq = \int 2q\,dq$$

Use integration by substitution.

$$\int \frac{1}{p(p+3)}\,dp = \int 2q\,dq$$

Express $\frac{1}{p(p+3)}$ as partial fractions.

$$\frac{1}{p(p+3)} = \frac{A}{p} + \frac{B}{p+3}$$

Put the RHS over one denominator.

$$= \frac{A(p+3)+pB}{p(p+3)}$$

Solve for A and B.

$$A = \frac{1}{3} \text{ and } B = -\frac{1}{3}$$

Write the integration.

$$\frac{1}{3}\int\left(\frac{1}{p} - \frac{1}{p+3}\right)dp = \int 2q\,dq$$

Simplify.

$$\frac{1}{3}\log_e\left|\frac{p}{p+3}\right| = q^2 + c$$

Assume $\frac{p}{p+3} > 0$ and rearrange.

$$\frac{p}{p+3} = Ae^{3q^2}, \text{ where } A = e^{3c}$$

Make the dependent variable p the subject.

$$p = Ape^{3q^2} + 3Ae^{3q^2}$$

Isolate p.

$$p\left(1 - Ae^{3q^2}\right) = 3Ae^{3q^2}$$

Solve for p.

$$p = \frac{3Ae^{3q^2}}{1 - Ae^{3q^2}} = \frac{3}{Be^{-3q^2} - 1}, \text{ where } B = \frac{1}{A}$$

Substitute in the known values.

$$10 = \frac{3}{Be^{-27} - 1}$$

Solve for B.

$$B = 6.916\ldots \times 10^{11}$$

Write the answer.

$$p = \frac{3}{Be^{-3q^2} - 1}, \text{ where } B \approx 6.92 \times 10^{11}$$

You could leave the solution in the A form with $A = 1.445... \times 10^{-12}$, but the B expression is considered 'more elegant' for calculation of values.

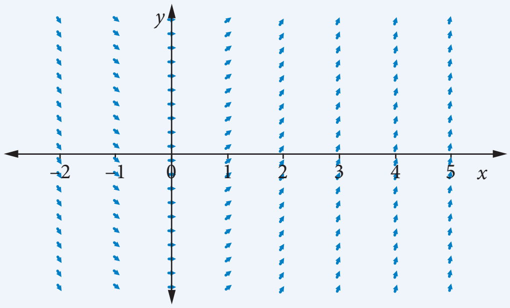
INVESTIGATION Gradient fields

Another way of visualising solutions to differential equations is to use a **gradient field**. A gradient field shows what a solution will look like by showing gradients of the function as short lines or arrows at multiple locations.

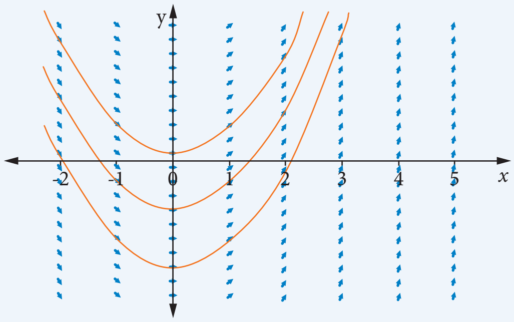
Consider the equation $\frac{dy}{dx} = 0.8x$. Find some values of the derivative.

x	−2	−1	0	1	2	3	4	5
$\frac{dy}{dx}$	−1.6	−0.8	0	0.8	1.6	2.4	3.2	4

The derivative is the slope of the tangent, so although you don't know what the value of y is, you can show the derivatives as short lines or arrows with the slope equal to the derivative.



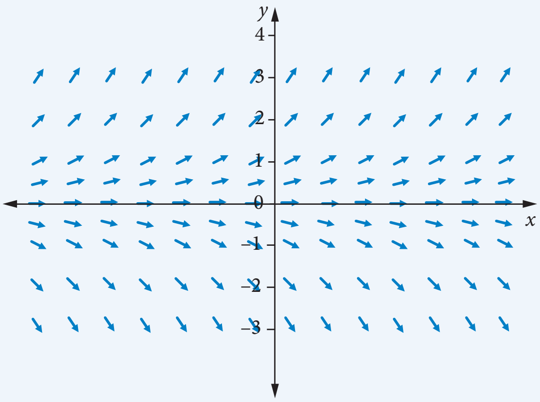
You can follow the arrows to join up possible curves.



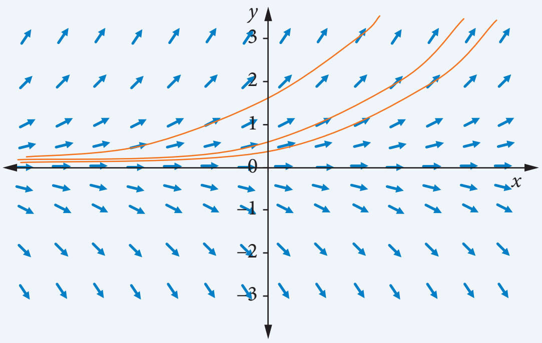
The curves that you get form a family of possible functions. It is hardly surprising that they look like parabolas.

Investigate the shapes that you get for the differential functions $\frac{dy}{dx} = 0.5 + 0.5x$ and $\frac{dy}{dx} = 0.6 - 0.6x$.

You can use the same technique, even when there is no independent variable on the RHS of the differential equation. Consider the differential equation $\frac{dy}{dx} = 0.5y$. You can draw a chart for $\frac{dy}{dx}$ against y-values and a gradient field by leaving the x-axis blank to make the arrows across the page the same slopes, as shown below.



Connecting these arrows in the same way as before gives a family of curves that look exponential. It is obvious that the x-axis is an asymptote.



Investigate the shapes you get for the differential functions $\frac{dy}{dx} = y + 0.5$ and $\frac{dy}{dx} = 1 - 0.6y$. If you are really adventurous, try $\frac{dy}{dx} = xy$.

EXERCISE 8.05 First order differential equations with separable variables

WS Differential equations

Concepts and techniques

1 Example 12 Solve $\dfrac{dy}{dx} = y(x-3)$.

2 Solve $\dfrac{dg}{dx} = (4g-7)(3x+1)$.

3 Example 13 Solve $\dfrac{dP}{dV} = P^2V$ if $P = 1$ when $V = 1$.

4 Given $v = 1$ at $t = 0$, solve $\dfrac{dv}{dt} = (4v-1)(3t+5)$.

5 Solve $\dfrac{dy}{dx} = x + xy$.

Reasoning and communication

6 Example 14 Solve $\dfrac{df}{dt} = f(3f+2)t$ if the initial value of f is 6.

7 Solve $\dfrac{dy}{dx} = \dfrac{(1+x^2)(1+y^2)}{xy}$

8 Solve $\dfrac{dy}{dx} + 2y = 3$ if $y = 10$ when $x = 0$

9 $\dfrac{dy}{dx} = xe^{x+y}$

10 $e^{2x+y} - 2e^{x-y}\dfrac{dy}{dx} = 0$

8.06 THE LOGISTIC EQUATION

The population of an animal introduced to a new environment will at first appear to grow exponentially. However, the population will asymptotically approach a maximum after some time because of the limitations of the resources in the environment. In this case, the rate of change of the population is proportional to two conflicting factors. Breeding will give a factor of kN, as the more there are, the greater the growth through breeding.

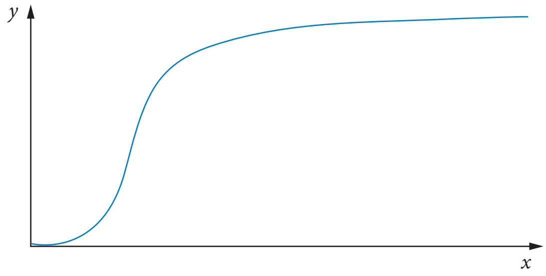
Alamy/Kunst and Scheidulin

As the population grows, the growth will also be proportional to $M - N$, where M is the maximum population that can be sustained by the environment. This means that the differential equation governing the rate of growth will be

$$\frac{dN}{dt} = kN(M - N)$$

For a continuous variable y with a maximum possible value this becomes

$$\frac{dy}{dt} = ky(M - y).$$



This is called the **logistic model**. The typical graph of such a model is shown on the right.

IMPORTANT

The **logistic model** of growth is given by $\frac{dy}{dt} = ky(M - y)$, where the rate of change of a variable is proportional to both the variable and the difference between a maximum value and the variable.

INVESTIGATION Epidemics

The spread of an infection through a population can be modelled by using N for the infected population, M for the susceptible population and k as a constant related to the infectiousness of the disease. The logistic model of the rate of change of the number of people infected will be given by $\frac{dN}{dt} = kN(M - N)$.

Suppose that a population of 100 000 people are not vaccinated (not immune) against a disease and that 100 people are initially infected, with $k = 0.000\,01$ for the time in weeks.

Plot the number infected over a period of 10 weeks.

Suppose that 20% of the population are vaccinated and k is reduced by 20% because contacts between infected and susceptible people are also reduced. Rework the model and comment on the changes.

Do this again with vaccination rates of 40%, 60%, 80% and 95% and comment on the results.

Example 15

In a town of 5000 people, 10 people have a new strain of cold. The spread of the cold through the town is given by the differential equation $\frac{dN}{dt} = 0.000\,24N(5000 - N)$, where t is in weeks and N is the number of people infected. Find an expression for the number of people infected and hence find the numbers infected after 2 and 4 weeks.

Solution

Write the differential equation.

$$\frac{dN}{dt} = 0.000\,24N(5000 - N)$$

Separate the variables.

$$\frac{1}{N(5000-N)}\frac{dN}{dt} = 0.000\,24$$

Integrate both sides.

$$\int \frac{1}{N(5000-N)}\frac{dN}{dt}\,dt = \int 0.000\,24\,dt$$

Use integration by substitution.

$$\int \frac{1}{N(5000-N)}\,dN = \int 0.000\,24\,dt$$

Write $\frac{1}{N(5000-N)}$ as partial fractions.

$$\frac{1}{N(5000-N)} = \frac{A}{N} + \frac{B}{5000-N}$$

Put the fractions on the RHS over a common denominator and equate denominators.

$$A(5000 - N) + BN = 1$$

Substitute $N = 0$ and $N = 5000$ to solve.

$$A = B = \frac{1}{5000} = 0.0002$$

Substitute into the integrals.

$$\int \left[\frac{1}{5000N} + \frac{1}{5000(5000-N)}\right] dN = \int 0.000\,24\,dt$$

Simplify.

$$\int \frac{1}{N}\,dN + \int \frac{1}{5000-N}\,dN = 5000 \times \int 0.000\,24\,dt$$

Perform the integrations.

$$\log_e(N) - \log_e(5000 - N) = 5000 \times 0.000\,24t + c$$

Simplify.

$$\log_e\left(\frac{N}{5000-N}\right) = 1.2t + c$$

Change to exponential form.

$$\frac{N}{5000-N} = Ae^{1.2t}$$

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Solve for N.	$N=\dfrac{5000Ae^{1.2t}}{1+Ae^{1.2t}}$
Multiply top and bottom by $e^{-1.2t}$.	$=\dfrac{5000Ae^{1.2t}}{1+Ae^{1.2t}}\times\dfrac{e^{-1.2t}}{e^{-1.2t}}$
	$=\dfrac{5000A}{A+e^{-1.2t}}$
Substitute in $t=0$ and $N=10$.	$10=\dfrac{5000\times A}{A+1}$
Solve for A.	$A=\dfrac{10}{5000-10}=\frac{1}{499}=0.002\ 004\ldots$
Write the model.	$N=\dfrac{5000A}{A+e^{-1.2t}}$, where $A=\dfrac{1}{499}$
Find the value at $t=2$.	$N(2)=108.065\ldots$
Find the value at $t=4$.	$N(4)=979.116\ldots$
Write the answer.	The number of people infected is given by $N=\dfrac{5000A}{A+e^{-1.2t}}$, where $A=\dfrac{1}{499}$, and the numbers infected were 108 after 2 weeks and 979 after 4 weeks.

The general solution for the logistic model can be derived in exactly the same way as the solution in Example **15**.

IMPORTANT

The general solution for the logistic model given by $\dfrac{dy}{dt}=ky(M-y)$ is

$$y=\frac{MA}{A+e^{-kMt}}\quad\text{or}\quad y=\frac{My_0}{y_0+(M-y_0)e^{-kMt}}$$

where $A=\dfrac{y_0}{M-y_0}$ and y_0 is the initial value.

Example 16

The maximum number of possums that can be supported by the environment of a national park is estimated to be 3000. When first counted, there were found to be 1200 possums, but after 4 months the number has grown to 1300. Use the logistic model to predict the number of possums after 1 year.

Corbis/Nature Connect

Solution

Write the logistic solution.

$$N = \frac{MN_0}{N_0 + (M - N_0)e^{-kMt}}$$

Substitute in $M = 3000$, $N_0 = 1200$, $t = 4$ and $N = 1300$.

$$1300 = \frac{3000 \times 1200}{1200 + (3000 - 1200)e^{-k \times 3000 \times 4}}$$

Solve for e^{-3000k}.

$$e^{-3000k} = 0.966\,281\ldots$$

Write the model.

$$N = \frac{3\,600\,000}{1200 + 1800(0.966...)^t}$$

Substitute $t = 12$.

$$N = \frac{3\,600\,000}{1200 + 1800(0.966...)^{12}}$$

$$= 1504.603\ldots$$

Write the answer.

After 12 months, the logistic model predicts that there will be 1504 possums in the park.

It is sometimes easier to find e^{-k} in a logistic model, as shown in Example **16**, to avoid some of the very small or large numbers that can occur.

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EXERCISE 8.06 The logistic equation

Reasoning and communication

1 Example 15 A new viral disease was found to spread according to the equation $\frac{dN}{dt} = kN(M - N)$, where M is the susceptible population, N is the number of people infected at time t months and $k = 1.5 \times 10^{-9}$. In March 2010, it was thought that only 100 people out of a population of 18 million were infected. Use the logistic model to find the number infected in:

a March 2011 b June 2012 c January 2017

2 Example 16 A remote island with no feral animals is used to set up a breeding population of bilbies. Initially, 10 bilbies are released on the island, which has an ideal environment and is thought to be able to sustain a population of 600 bilbies. After 2 years, the population has grown to 30 bilbies. Use the logistic model to find:

Auscape/Glen Threlfo

a when the population will reach 200

b the number of bilbies that could be taken back to the mainland in each following year while maintaining the population at 200.

3 A poison is dissolved in honey and set as ant baits to get rid of a nest of fire ants, estimated to have 10 000 ants. When the bait is set, 10 ants are observed to be killed immediately. The nest is observed after 1 week, when the baits are renewed, and the activity is seen to have reduced by 1%. Use the logistic model for the number of ants killed to find how long the baits must be used to ensure that the nest is exterminated. It is considered that the nest will be exterminated when the population is below 50, as this is the number needed to maintain the queen.

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4 A phone company knows that about 40% of mobile phone owners will buy a phone that has new features whether or not they are satisfied with their current phone. Other mobile users are unlikely to change unless there is something wrong with their existing phone. Past experience suggests that initial sales will be about 1% of the market, with market penetration of 5% after 4 weeks. They estimate that they have a period of 9 weeks before their competitors bring out a phone with the new features. How much of the market can they expect to capture before this happens?

5 An area in the Channel Country can support about 3000 rabbits in dry times, but after a flood it can support about 30 000. The Department of Primary Industries (DPI) is interested in predicting the population in the 12 months following a flood. It is estimated that, if no control measures are taken, the population will grow to 5000 in the first month; but if normal control measures (trapping, shooting, poison) are taken, the population will grow to 3500 in the first month.

a Use the logistic model to determine the population each month after a flood with no control measures.

b Use the model to find the population each month with normal control measures.

c Draw a graph, showing both situations on the same axes.

d The DPI estimates that, if the population remains below 10 000 in the year following a flood, no serious damage will occur. Comment on your results.

6 It has been said that the maximum population that Australia can support is 50 million. The population reached 15 million in 1981 and 21 million in 2008. Use the logistic model to predict the population in:

a 2015 b 2020 c 2030 d 2050

7 The logistic model can be used to predict the death rates from a toxin. The LD-50 quantity is the amount that must be given to kill 50% of the subjects. A particular poison is tested on laboratory mice for effectiveness as a control for house mice. The LD levels were found to be 2 g for LD-50 and 3 g for LD-70. The maximum level (LD-100) is theoretically unobtainable.

a What quantity of toxin would be needed for:

i LD-90? ii LD-95? iii LD-99?

b What percentage of mice would have died with no poison?

8 A 'whisper campaign' regarding the financial affairs of an election candidate is started 4 days before the election. There are 33 420 voters in the electorate and the campaign is started by about 50 voters loyal to another candidate. After 1 day, it is estimated that 400 voters are aware of the rumour. How many people will hear the rumour before the election?

9 The completion of a particular chemical reaction involving a catalyst follows the logistic model. The reaction between molecules of the reactants occurs when they are adsorbed onto the catalyst sufficiently close to one another to react. As the reaction proceeds, this is less and less likely to occur. When first measured, 5% of the reaction has taken place; and after 5 minutes, another 5% has taken place. How long is it until the reaction proceeds to completion (99% reacted)?

10 A new service providing mini-gardens planted with vegetables was planned for inner Melbourne residents with small outdoor spaces. The service was expected to serve a maximum of 2000 residents in the suburbs where it was to be offered. A survey in Fitzroy found that 30 residents would be interested in taking the service immediately with another 10 in a fortnight's time. The rent for each garden supplied is to be \$20 per week. Use the logistic model to determine how long it would take to reach the operational break-even point of \$2000 a week in rents.

8.07 FORCE, MOMENTUM AND MOTION

Everybody has an intuitive understanding that movement does not happen without a reason. To make something move, you push or pull it in some way. You also know that 'light' objects respond more to the same push than 'heavy' objects. These ideas were formalised by Sir Isaac Newton (1642–1727). His first law of motion is essentially a definition of **force** as that 'something' that causes motion or changes a motion.

IMPORTANT

Newton's first law of motion

Unless acted on by a resultant **force**, a body remains at rest or in uniform motion in a straight line.

Uniform motion means speed, so the law says that objects will either remain at rest or keep moving with constant velocity unless they are acted upon by a force. The term 'resultant' just means that balanced forces will not change an object's motion. In this section, the words *body* and *particle* will be used interchangeably to describe an object. Forces are presumed to act through the centre of the object. The concept of 'light' and 'heavy' objects is encapsulated in the idea of inertial **mass**. If the same force produces twice the acceleration on one object compared to another, then you say that the mass of the object that accelerates faster is half that of the slower object.

IMPORTANT

Newton's second law of motion

The acceleration of a body is directly proportional to the resultant force and inversely proportional to the **mass** of the object. The acceleration is in the direction of the resultant force. Units are chosen so that this law becomes $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ and $\mathbf{a}$ are vectors and m is a scalar.

The second law is really a definition of mass, but it does not define a unit. The international unit of mass is defined as the mass of the international prototype of the kilogram. This is made of platinum-iridium and kept in Paris under strict conditions. Copies of it are kept in National Standards Laboratories around the world. When several forces are acting at the same time, the motion is determined by the vector sum of the forces.

The SI unit of force is the newton (N), which is the force needed to accelerate 1 kg at 1 m/s/s.

Example 17

The vent in a hot-air balloon of mass 600 kg is opened to begin descent from a stationary position 120 m above level ground. Gravity is exerting a force of 5880 N downwards, the lift provided by the hot air is 5000 N upwards and a wind springs up and drives it eastwards with a force of 300 N.

a What is the acceleration of the balloon?

b How long will it take to reach the ground?

c At what speed will it hit the ground?

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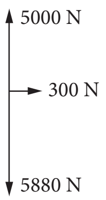
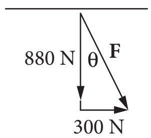
Solution

a Draw a labelled diagram of the forces.



Find the resultant downward force. — Downward force = 880 N

Draw a labelled diagram to find the total force **F**.



Find the magnitude of **F**.

$$F = \sqrt{880^2 + 300^2}$$
$$= 929.7\ldots \text{ N}$$

Find the angle that **F** makes with the vertical.

$$\tan(\theta) = \frac{300}{880}$$

Solve for θ. — $\theta = 18.82\ldots°$

Write Newton's second law. — $F = ma$

Substitute and find a. — $929.7\ldots = 600a$, so $a = 1.549\ldots \text{ m/s}^2$

State the answer. — The balloon will accelerate downwards at about 19° to the vertical at about 1.55 m/s^2.

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b	Find the vertical component of the acceleration.	$a_v = a\cos(\theta)$
	Write the equation for the vertical velocity.	$a = \frac{d\mathbf{v}}{dt}$, so $v = at + c$
	Use the stationary start.	At $t = 0$, $v = 0$, so $c = 0$
	Write the equation for the height.	$\frac{dh}{dt} = a_v\, t$, so $h = \frac{1}{2}a_v t^2 + c$
	Use $h = 120$ at $t = 0$ to find c.	$c = 120$
	Write the equation to find t when $h = 0$.	$0 = \frac{1}{2}a_v t^2 + 120$
	Find t (with acceleration downwards).	$t = \sqrt{\dfrac{-120}{\frac{1}{2}\times(-1.549\ldots)\times\cos(18.82\ldots°)}}$ $= 12.79\ldots \text{s}$
	Write the answer.	The balloon takes about 12.8 s to reach the ground.
c	Use the velocity equation.	$v = at$
	Substitute in the values.	$= -1.549\ldots \times 12.79\ldots$ $= -19.8\ldots$ m/s
	Write the answer.	The balloon will hit the ground at about 20 m/s.

A prudent hot-air balloon operator might close the vent some more in Example **17** as 20 m/s is about 70 km/h, which is perhaps a little fast for his passengers.

In addition to defining mass, it is useful to define **momentum**, as this also gives us an idea of the kinds of forces needed to change motion.

IMPORTANT

The **momentum**, **p**, of a body is defined as the product of its mass and velocity. $\mathbf{p} = m\mathbf{a}$ is a vector. Newton's second law of motion can also be expressed as $\mathbf{F} = \frac{d\mathbf{p}}{dt}$

The equivalence is easily shown as $\frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}$.

Newton completed his three laws of motion with the following.

IMPORTANT

Newton's third law of motion

For every action there is an equal and opposite reaction.

This means that forces occur in opposite pairs. It is an expression of the fact that you cannot push something without something to push against. It is also equivalent to saying that the total momentum of a system remains constant, or that momentum is conserved.

Example 18

An eight-ball with a mass of 0.12 kg struck the cushion directly on at 3 m/s and rebounded at 2 m/s. The collision with the cushion took 2 ms. What was the average force that the ball exerted on the cushion?

Solution

Find the change of momentum.

$$\Delta \mathbf{p} = 0.12 \times 2 - 0.12 \times (-3) \text{ kg m/s}$$
$$= 0.6 \text{ kg m/s}$$

Find the average rate of change of momentum.

$$\frac{\Delta \mathbf{p}}{\Delta t} = \frac{0.6}{0.002} = 300 \text{ N}$$

Write the answer.

The average force exerted on the cushion was 300 N.

The ball in Example **18** exerted a force on the cushion, which is part of the table, so why doesn't the table move in this situation? The reason is that the Earth-ball-table system is essentially a closed system. When the player strikes the ball, a force is exerted on the ball and an equal and opposite force by the player's feet on the Earth, so the Earth moves back a miniscule amount. When the ball hits the cushion, it moves back the other way. Eventually the whole system ends up back in the same place, except for a slight change in the position of the Earth-table-player to compensate for the change of the ball's position. However, for someone outside the Earth, there is no overall change as they cancel out.

EXERCISE 8.07 Force, momentum and motion

Concepts and techniques

1 Example 17 A body of mass 20 kg is acted on by two forces, $\mathbf{P} = 12\mathbf{i} - 4\mathbf{j}$ and $\mathbf{Q} = 20\mathbf{i} + 32\mathbf{j}$, where both forces are in newtons. Calculate the magnitude and direction of the acceleration.

2 A block of mass 3 kg is acted on by a force $\mathbf{R} = 2\mathbf{i} + 5\mathbf{j}$ N. Calculate the magnitude and direction of the acceleration.

3 A uniform force acts on a box with a mass of 5 kg and causes the box to accelerate at $(4\mathbf{i} + 8\mathbf{j})$ m s^{-2}. Find the magnitude and direction of the force acting on the box.

4 A force of 19.6 N acts on a brick of mass 2.4 kg for 5 s. Find:

a the acceleration

b the speed after 5 s

c the distance travelled in this time.

5 Example 18 A force of 50 newtons acts on an object of mass 8 kg, that is initially at rest, for a period of 4 seconds.

a What is the change of momentum?

b What is the speed of the object after the force ceases to act?

6 A car of mass 1.1 tonnes (with passengers) travelling in a straight line at 100 km/h slows down to 46 km/h in a period of 4 seconds. What force was applied by the brakes?

7 A force of 200 newtons acts in an easterly direction on a mass of 20 kg moving north at 10 m/s for 1.5 seconds.

a Find the change of momentum of the mass.

b Find the new velocity of the mass.

Reasoning and communication

8 A wire is stretched between two poles at a distance of 50 m apart. The tension on the wire is 400 N. A bird lands in the middle of the wire, making it sag a distance of 20 mm in the middle so that the tension increases to 1500 N. The original and new positions of the wire can be considered to be an isosceles triangle. What force does the weight of the bird exert on the wire? [Hint: the bird's weight is balanced by the extra tension acting in both directions]

9 A coal train weighing 23 000 tonnes is moving east at 54 km/h. Without changing speed, the train negotiates a curve over 2 minutes to moving in the ENE direction. Assuming that the force was constant, what force was required to change the train's direction?

10 Three identical tennis balls of mass 57 g are squashed together on flat concrete and held in place with a piece of string. The string is released by burning it through and the tennis balls move away from each other across the concrete. One of the balls goes off at 5 m/s northwards and a second one goes off at 6.5 m/s on a bearing of 120°.

a What is the change in momentum of the first ball from the original position?

b What is the change in momentum of the second ball?

c At what speed and in what direction does the third ball move off in?

8.08 MOTION IN A STRAIGHT LINE WITH CONSTANT FORCE

If motion is in a straight line, you can use the sign (+/–) of the acceleration a, velocity v, and position or displacement s for the direction. You do not need to write these as the vectors **a**, **v** or **s**.

There are a number of situations where a constant force is applied to an object, such as the force of gravity near the surface of the Earth. In this case, the acceleration is constant, so you can write $\frac{dv}{dt} = a$, where a is the (constant) acceleration.

Then you get $v = at + c$. For an initial velocity of v (when $t = 0$) you get $v = u + at$.

You can then write $s = \int (u+at)dt = ut + \frac{1}{2}at^2 + c$.

If the initial position is s_0, you get $s = s_0 + ut + \frac{1}{2}at^2$.

If $s_0 = 0$ or s is considered as displacement from the initial position, then you get $s = ut + \frac{1}{2}at^2$.

You can rewrite $v = u + at$ as $a = \frac{v-u}{t}$ and substitute it in $s = ut + \frac{1}{2}at^2$ to get

$$s = ut + \frac{1}{2} \times \frac{v-u}{t} \times t^2$$

$$= ut + \frac{1}{2}(v-u)t$$

$$s = \frac{1}{2}(v+u)t \text{ or } s = \left(\frac{v+u}{2}\right)t.$$

Then you can multiply $a = \frac{v-u}{t}$ and $s = \left(\frac{v+u}{2}\right)t$ to get $as = \frac{v-u}{t} \times \frac{v+u}{2} \times t$.

The ts cancel to give $as = \frac{v^2 - u^2}{2}$ or $v^2 = u^2 + 2as$.

IMPORTANT

Equations for motion in a straight line with constant force

For constant acceleration a, velocity v, initial velocity u, time t and displacement s, you can write each of the following equations for motion in a straight line.

$$v = u + at \qquad s = ut + \frac{1}{2}at^2$$

$$s = \left(\frac{v+u}{2}\right)t \qquad v^2 = u^2 + 2as$$

It is useful to be able to change from m/s to km/h and vice versa. The conversion can be worked out as 1 m/s = 3600 m/h = 3.6 km/h, so 1 m/s = 3.6 km/h.

Example 19

A car travelling at 80 km h^{-1} is brought to rest in a distance of 50 m. The brakes are applied uniformly.

a What is the deceleration experienced by the car?

b Find the time taken for the car to stop.

Solution

a Change the speed to m/s. — 80 km/h = 80 ÷ 3.6 m/s

Calculate the answer. — = 22.222… m/s

Write the equation with speed, distance and acceleration. — $v^2 = u^2 + 2as$

Substitute values. — $0^2 = (22.222\ldots)^2 + 2a \times 50$

Solve for a. — $a = -4.938\ldots \text{ m/s}^2$

Write the answer. — The deceleration is about 4.94 m/s^2.

b Choose the best equation. — $s = \left(\frac{v+u}{2}\right)t$

Substitute in the values. — $50 = \left(\frac{0+22.22\ldots}{2}\right)t$

Solve for t. — $t = 4.5$ s

Write the answer. — The car takes 4.5 s to stop.

The acceleration of gravity near the surface of the Earth is generally taken to be –9.8 m/s^2, with the negative sign indicating the direction down.

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Example 20

A stone is thrown vertically upward and is seen passing a point 10 m high on the face of a building 3 s after being thrown. What was the initial velocity of the stone?

Solution

Write the acceleration.	$a = -9.8$ m/s
Choose the equation with s, t, a and u.	$s = ut + \frac{1}{2}at^2$
Substitute in the values.	$10 = u \times 3 + \frac{1}{2} \times (-9.8) \times 3^3$
Solve for u.	$u = 18.033\ldots$
Write the answer.	The initial velocity was about 18 m/s.

There is often more than one way to solve a problem.

Example 21

A cyclist travelling at a constant speed of 18 km/h passes a truck just as it begins to move in the same direction. The truck accelerates uniformly at 0.4 m/s^2 for 20 s and then continues with uniform speed. How far will the truck travel before reaching the cyclist?

Solution

Change the cyclist's speed to m/s.	Cyclist's speed = $18 \div 3.6 = 5$ m/s
Find the cyclist's position after 20 s.	$s_C = 20 \times 5 = 100$ m
Find the truck's position after 20 s.	$s_T = ut + \frac{1}{2}at^2$
Substitute in the values.	$= 0 + \frac{1}{2} \times 0.4 \times 20^2$ $= 80$ m
Find their separation.	Distance between them = $100 - 80 = 20$ m
Find the truck's speed.	Truck's speed = $u + at$ $= 0 + 0.4 \times 20$
Calculate the value.	$= 8$ m/s
You can use relative speeds.	**Method 1**
Find the relative speed.	Speed of truck relative to cyclist = $8 - 5$ $= 3$ m/s
Find the time to catch up.	Time to catch up = $\frac{20}{3} = 6\frac{2}{3}$ s
You can also use time.	**Method 2**
Write the positions t seconds later.	Cyclist's position = $20 + 5t$; Truck = $8t$
Find when they coincide.	$20 + 5t = 8t$

Solve for *t*.	$t = 6\frac{2}{3}$ s
Find the position of the truck.	Truck's position $= 80 + 6\frac{2}{3} \times 8$ $= 133\frac{1}{3}$ m
State the answer.	The truck travels $133\frac{1}{3}$ m altogether before reaching the cyclist.

EXERCISE 8.08 Motion in a straight line with constant force

Concepts and techniques

1 Example 19 A car travelling at 54 km/h comes to rest with uniform retardation in 5 s. Calculate the acceleration and the distance travelled in the time taken for the car to be brought to rest.

2 A motor vehicle increases its speed from 18 km/h to 72 km/h in a distance of 50 m under uniform acceleration. Calculate:

a the acceleration

b the speed of the vehicle when 25 m has been covered.

3 A particle with an initial velocity of 100 m/s experiences a retardation of 2 m/s^2. How long will it take to bring the particle to rest, and how far will it travel in this time?

4 A monorail train begins from rest and travels with a uniform acceleration of 1.2 m s^{-2}. What will be the speed of the monorail after 30 s, and how far will it have travelled?

5 An object moves with uniform acceleration for 3 s, in which time it travels 27 m. It then moves with uniform velocity for the next 5 s, covering 60 m. Calculate:

a the initial velocity

b the acceleration in the first part of its journey.

6 Example 20 A carriage of a train moves in a straight line with uniform acceleration.

a How far does the carriage travel in 12 s if it moves 48 m in the first 6 s of its motion and 32 m in the last 2 s?

b What is the initial velocity?

Reasoning and communication

7 Example 21 Cyclist *X*, riding at 16 km/h is overtaken and passed by cyclist *Y*, who is travelling at 20 km/h. At this point, *X* immediately increases speed with uniform acceleration. Show that *X* will catch *Y* when she reaches a speed of 24 km/h.

8 A roller-coaster begins from rest and moves with uniform acceleration. It travels 9.5 m in the tenth second after starting its trip. Find:

a the acceleration

b the total distance covered during the first 5 s of motion.

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9 A cage carrying workers down a mine shaft completes the 675 m descent in 45 s. During the first quarter of the time, the cage is uniformly accelerated, while in the last quarter it is uniformly retarded, the acceleration and retardation being equal in magnitude. Calculate the uniform speed of the cage in the central portion of its descent.

10 A ball thrown upwards from the top of a cliff took 2 seconds longer to fall to the bottom of the cliff than it did to reach the top of its 8 second flight. How high is the cliff?

11 A rower has a stronger right arm so the force exerted by his right oar on the boat is an average of 60 N in a direction 20° to the left of the forward direction, while the force exerted by his left oar is 50 N at 20° to the right of the forward direction. The friction with the water is 50 N towards the rear. The rower and boat together have a mass of 280 kg.

a What is the acceleration of the boat?

b What is its speed a minute after starting?

c How far off course is the boat a minute after starting?

12 The force produced by the wind on a yacht of mass 1.4 t as it starts tacking into the wind is 60 000 N at 120° to the direction of travel.

a What force must be produced by the keel and rudder for it to move forward?

b The drag on the yacht from the water is 20 000 N. What is the initial acceleration?

8.09 MOTION IN A STRAIGHT LINE WITH VARIABLE FORCES

Most real forces are not constant, so the accelerations they produce are not constant either. Some forces depend on the velocity of an object, some depend on distance and some depend on time. Those that depend on time are relatively easy to deal with using integration.

Example 22

A 5 kg body initially at rest is acted on by a force that has a magnitude that varies with time according to the expression $40t - 6t^2$, where t is in seconds. If the direction of the force remains constant, calculate the velocity of the body after 5 s.

Solution

Use Newton's second law.	$F = ma$
Substitute in the values.	$40t - 6t^2 = 5a$
Find the acceleration.	$a = 8t - 1.2t^2$
Integrate to find the velocity.	$v = \int a\,dt = \int (8t - 1.2t^2)\,dt$
Simplify.	$v = 4t^2 - 0.4t^3 + c$
Use $v = 0$ at $t = 0$.	$c = 0$ so $v = 4t^2 - 0.4t^3$
Substitute in $t = 5$.	$v = 4 \times 5^2 - 0.4 \times 5^3$
Calculate v.	$= 50$ m/s
Write the answer.	After 5 s, the velocity is 50 m/s.

In cases where the force and acceleration depend on distance or velocity, it is useful to have calculus expressions that do not involve time.

Using the chain rule, you can write

$$a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$$

But $v = \frac{dx}{dt}$, so $a = v\frac{dv}{dx}$. Using the chain rule in reverse, you get

$$v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dv} \times \frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

IMPORTANT

The acceleration of an object may be expressed in any of the forms

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

Example 23

A particle moving in a straight line has an acceleration of $5 - 2x$, where x is the position at time t. The velocity at $x = 3$ is 4. Find an expression for the velocity.

Solution

Write the appropriate expression for a.

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

Use integration.

$$\begin{aligned}\frac{1}{2}v^2 &= \int a\,dx \\ &= \int (5 - 2x)dx \\ &= 5x - x^2 + c\end{aligned}$$

Substitute in the values to find c.

$$\frac{1}{2} \times 4^2 = 5 \times 3 - 3^2 + c$$

Solve for c.

$$c = 2$$

Write the expression for v.

$$\frac{1}{2}v^2 = 5x - x^2 + 2$$

Find v.

$$v = \sqrt{10x - 2x^2 + 4}$$

Write the answer.

The velocity is given by $v = \sqrt{10x - 2x^2 + 4}$

Friction in a medium like air depends on the velocity of the object. It is a quadratic relationship, but for larger objects it can be considered as linear, while for very small objects it is proportional to v^2. As speed increases, the total force decreases because the air friction gets closer in magnitude to the forward force. Planes can produce only a limited forward force from their engines, so eventually the drag is equal to the forward force and the plane stops accelerating. Objects falling under the influence of gravity will eventually reach a **terminal velocity**, where the force of gravity is balanced by the force due to air friction.

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Example 24

A person free-falling has air friction of $F = kv$ acting to retard their motion. For an 85 kg person of normal body shape, the value of k is about 16. What is the terminal velocity?

Solution

Use Newton's 2nd law for gravity.	Force due to gravity $= mg$, where $g = 9.8$ m/s^2
Write the total force with an open parachute.	Total force $= mg - kv$
Use Newton's 2nd law for the downward acceleration.	$mg - kv = ma$
Solve for a.	$a = g - \frac{kv}{m}$
Write as a derivative.	$\frac{dv}{dt} = g - \frac{kv}{m}$
	$= -\frac{k}{m}\left(v - \frac{mg}{k}\right)$
Isolate v.	$\frac{1}{\left(v - \frac{mg}{k}\right)}\frac{dv}{dt} = -\frac{k}{m}$
Integrate both sides with respect to time.	$\int \frac{1}{\left(v - \frac{mg}{k}\right)}\frac{dv}{dt}\,dt = \int -\frac{k}{m}\,dt$
Use integration by substitution.	$\int \frac{1}{\left(v - \frac{mg}{k}\right)}\,dv = \int -\frac{k}{m}\,dt$
Perform the integrations.	$\log_e\left(v - \frac{mg}{k}\right) = -\frac{k}{m}t + c$
Express as an exponential.	$v - \frac{mg}{k} = Ae^{-\frac{kt}{m}}$
Simplify.	$v = \frac{mg}{k} + Ae^{-\frac{kt}{m}} = \frac{m}{k}\left(g + Be^{-\frac{kt}{m}}\right)$
	where $B = \frac{kA}{m}$
The terminal velocity is the value as $t \to \infty$.	As $t \to \infty$, $Be^{-\frac{kt}{m}} \to 0$
Write the expression for terminal velocity.	Terminal velocity $= \frac{mg}{k}$
Substitute in the values.	$= \frac{85 \times 9.8}{16}$
	$= 52.0625$
Write the answer.	The person's terminal velocity is about 52 m/s.

Notice in Example **24** that the constant for terminal velocity depends on the value of $\frac{m}{k}$, so for a particular body shape the terminal velocity will be about the same.

EXERCISE 8.09 Motion in a straight line with variable forces

Concepts and techniques

WS Motion with variable forces

1 Example 22 In each of the following situations, m = mass (kg), u = initial velocity (m s^{-1}) and F = force (N). Use this information to find an expression for velocity as a function of time, $v(t)$.

a $m = 1$, $u = 0$ and $F = 2t - 5t^2$

b $m = 3$, $u = 0$ and $F = \sin(3t - 2)$

c $m = 8$, $u = 3$ and $F = 3e^{0.4t+5}$

d $m = 2$, $u = 7$ and $F = 3v$

2 Example 23 In each of the following situations, m = mass (kg), u = initial velocity (m s^{-1}) and F = force (N). Use this to find an expression for velocity as a function of position, $v(x)$, if $x = 0$ when $t = 0$.

a $m = 2$, $u = 4$ and $F = \cos(x)$

b $m = 5$, $u = 2$ and $F = 2x - 7$

c $m = 4$, $u = 3$ and $F = 2v$

d $m = 1$, $u = 2$ and $F = 4v - v^2$

e $m = 3$, $u = -2$ and $F = \dfrac{1}{x+1}$

3 An object of mass 2 kg moves from rest under the action of a force with magnitude after t s given by $40 - 20t$ N. When the object reaches a speed of 20 m s^{-1}, the force is removed and the object continues with constant speed for 4 s. It is then brought to rest by a constant force of 60 N. Calculate the total distance travelled.

4 A body of mass 5 kg is acted on by a force in a straight line. The size of the force after t s is given by $10(2t - 3t^2)$ N. If the body has an initial velocity of 3 m s^{-1} in the same direction as the force, calculate its velocity after 4 s.

5 Example 24 An object of mass 4 kg is projected vertically upward against air resistance that is proportional to the square of the speed and equal to 20 N when the speed is 10 m s^{-1}. Find the greatest height reached by the object if its initial velocity is 20 m s^{-1}.

6 An elastic ball of mass 2 kg is dropped from a height of 50 m and experiences air resistance equal to kv^2. The ball reaches the ground with a speed of 30 m s^{-1}.

a Determine the value of k in the expression for air resistance.

b How long after being dropped will the ball first reach the ground?

c If the ball rebounds vertically with a speed of 25 m s^{-1}, what is the maximum height that it will reach?

Reasoning and communication

7 A 5 kg body with an initial velocity of 14 m s^{-1} travels in a horizontal plane. The only force acting on the particle is the friction between it and the surface on which it moves. The magnitude of the frictional force is given by $F = 0.2v + 5$ N. Determine:

a an expression for velocity in terms of time

b the distance travelled by the body before it comes to rest.

8 A particle of mass m moving in a horizontal direction is subjected to air resistance given by $r = kv$, where k is a constant and v is the speed at time t and position x. Determine:

a an expression for velocity in terms of time if the initial velocity is u

b an expression for the displacement, x, at time t

c the limiting distance travelled by the particle.

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9 A particle is projected vertically upward with velocity u in a medium with resistance to motion equal to kv^2, where k is a constant and v is the velocity at time t. Find an expression for the greatest height reached by the particle.

10 A body of mass m is projected vertically upward with a velocity of u. If the air resistance is given by $r = kv$, where k is a constant and v is the speed of the body, find expressions for the greatest height h reached and the time t_h taken to reach that height.

11 A 1 kg particle falls from rest in a medium where the resistance to motion is given by $r = kv^2$, where k is a constant and v is the speed.

a Show that the distance, x, that the particle falls down when its velocity is v, is given by $x = \frac{1}{2k}\log_e\left(\frac{g}{g - kv^2}\right)$.

b Find the distance that the particle falls down when it reaches half of its terminal velocity.

8.10 SIMPLE HARMONIC MOTION

Simple harmonic motion is used to model vibrations. The horizontal movement of a pendulum and similar objects is also modelled by simple harmonic motion. The assumptions for simple harmonic motion are as follows.

IMPORTANT

The assumptions for simple harmonic motion are:

1 the motion is in a straight line

2 the force is given by $\mathbf{F} = -k\mathbf{x}$, where $\mathbf{x}$ is the displacement of the particle from a fixed point called the **point of equilibrium**

3 there is no friction

The vector equation for the force means that the force is acting to restore the particle to the point of equilibrium. You can treat the vectors as simple positive and negative numbers relative to the point of equilibrium.

Using $F = ma$, you can write $a = -\frac{k}{m}x$. This is also written as $a = -\omega^2 x$ for convenience.

Then you can write $\frac{d\frac{1}{2}v^2}{dx} = -\omega^2 x$.

Integrating both sides gives $\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c$, so $v^2 = -\omega^2 x^2 + 2c$.

In order to simplify the formulas, $2c$ is written as $a^2\omega^2$, giving $v^2 = a^2\omega^2 - \omega^2 x^2$ with the new constant a. Note that this a is *not* the acceleration.

Simplifying gives $v = \pm\sqrt{a^2 - x^2}$.

Rearrangement gives $\frac{1}{\sqrt{a^2 - x^2}}\frac{dx}{dt} = \pm\omega$.

You can integrate both sides to get $\int \frac{1}{\sqrt{a^2 - x^2}}\frac{dx}{dt}dt = \pm\omega t + c$

Now you can use integration by substitution and $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right)$ to get

$\arcsin\left(\frac{x}{a}\right) = \pm\omega t + c$ so $x = a \sin(\pm\omega t + c)$. Using $-\omega$ instead will give cos instead of sin.

It is usual to write the phase shift as α and choose the positive value of ω, giving $x = a \sin(\omega t + \alpha)$.

a is the **amplitude** of the motion, ω is called the **angular velocity** and the time for one complete motion is given by the **period** $T = \frac{2\pi}{\omega}$. The number of complete cycles in unit time is called the **frequency** and is given by $\frac{1}{T}$. In order to avoid confusion with amplitude, you normally use the 'dot' notation for the acceleration in simple harmonic motion.

IMPORTANT

The equations for simple harmonic motion with **angular velocity** ω, **amplitude** a, **period** T, **frequency** f and **phase** α are:

Acceleration: $\ddot{x} = -\omega^2 x = -a\omega^2 \sin(\omega t + \alpha)$ or $-a\omega^2 \cos(\omega t + \alpha)$

Velocity: $v = \dot{x} = \pm\omega\sqrt{a^2 - x^2} = -a\omega \sin(\omega t + \alpha)$ or $a\omega \cos(\omega t + \alpha)$

Displacement: $x = a \cos(\omega t + \alpha)$ or $x = a \sin(\omega t + \alpha)$

Period and frequency: $T = \frac{2\pi}{\omega} = \frac{1}{f}$, $\omega = 2\pi f$

Example 25

A particle oscillates 1.2 m either side of a central position with simple harmonic motion.

The period of the motion is 8 s.

a What is the maximum speed?

b Find the speed when the particle is 0.8 m from the central position.

c The particle started its motion from the extreme position. What is the acceleration 0.3 s later?

Solution

a Write the formula for the period. $T = \frac{2\pi}{\omega}$

Find ω. $\omega = \frac{2\pi}{T} = 0.25\pi \approx 0.785$ radians/s

Find the value at cos (0). Maximum speed $= a\omega = 1.2 \times 0.25\pi = 0.3\pi \approx 0.942$ m/s

Write the answer. The maximum speed is about 0.942 m/s.

b Use $v = \pm\omega\sqrt{a^2 - x^2}$ Speed $= \omega\sqrt{a^2 - x^2}$

Substitute in the values. $= 0.3\pi \times \sqrt{1.2^2 - 0.8^2} \approx 0.843$ m/s

Write the answer. The speed at $x = 0.8$ is about 0.843 m/s.

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c Use $x = a\cos\ldots$ for an extreme starting point.	Acceleration $= -a\omega^2 \cos(\omega t + \alpha)$
Substitute the values.	$= 1.2 \times (0.25\pi)^2 \cos(0.25\pi \times 0.3)$
Calculate the answer.	≈ 0.720 m/s^2
Write the answer.	The acceleration at 0.3 s is about 0.72 m/s^2.

Make sure that you set your calculator to radians when calculating results in simple harmonic motion.

Example 26

The end of a spring vibrates 20 times a second. The spring compresses to 16 cm and lengthens to 20 cm while vibrating. Find the natural length of the spring and the maximum acceleration of the end.

Solution

Find the equilibrium point.	Centre of motion has length $\frac{1}{2}(16 + 20) = 18$ cm.
Find the angular velocity.	$\omega = 2\pi f = 40\pi \approx 125.7$ radians/s.
Use $a\omega^2 \sin(\omega t + \alpha)$.	Maximum acceleration $= a\omega^2$
	$= 3200\pi^2$ cm/s^2
Calculate an approximate answer.	≈ 315.8 m/s^2

EXERCISE 8.10 Simple harmonic motion

Concepts and techniques

1 Examples 25, 26 A spring makes 12 vibrations in 3 s. Calculate the period and frequency of the vibration.

2 A body in simple harmonic motion makes 90 vibrations per minute. Calculate its period and frequency.

3 A mass vibrates in simple harmonic motion with an amplitude of 75 cm and a period of 3 s. Calculate:
- **a** the frequency
- **b** the maximum velocity (the velocity when displacement $x = 0$)
- **c** the velocity when $x = 50$ cm
- **d** the maximum acceleration
- **e** the acceleration when $x = 50$ cm.

4 A mass vibrates up and down along a straight line, 20 cm long, in simple harmonic motion with a period of 4 s. Calculate:
- **a** the amplitude of the vibration
- **b** the velocity and acceleration of the body at the midpoint of its path
- **c** the velocity and acceleration of the body at the upper end of its path
- **d** the velocity and acceleration of the body when its displacement from the centre is 4 cm.

Reasoning and communication

5 A particle moves in a straight line with simple harmonic motion. Calculate the time taken to undergo a complete oscillation if:

a the acceleration at a distance of 1.2 m is 2.4 m s^{-2}

b the acceleration at a distance of 20 cm is 3.2 m s^{-2}.

6 A particle moving with simple harmonic motion has an amplitude of 1.5 m. If the acceleration at a distance of 60 cm from the equilibrium position is 1.2 m s^{-2}, find the velocity when the particle is:

a in the equilibrium position

b 1.2 m from its equilibrium position.

7 A particle moving with simple harmonic motion passes through two points P and Q that are 56 cm apart with the same velocity. It takes the particle 2 s to move from P to Q and then another 2 s to return to Q. Calculate the period and amplitude of the oscillation.

8 A particle undergoing simple harmonic motion completes one full oscillation every 2 s. If the amplitude of the oscillation is 90 cm, calculate the maximum velocity and the maximum acceleration.

9 An object in simple harmonic motion performs 150 complete oscillations every minute. If the greatest acceleration achieved is 3 m s^{-2}, calculate the greatest velocity and the distance between the extreme points of the motion.

10 A particle moving with simple harmonic motion performs 45 complete oscillations per minute. The velocity at a point 2.5 cm from the central point is 30 cm s^{-1}.

a Calculate the greatest distance reached from the central point.

b If P and Q are two points positioned 2.5 cm and 4 cm respectively from the centre of motion, find the time taken in moving from P to Q.

11 The speed of a particle is given by $v = 2\sqrt{5 + 4x - x^2}$ m s^{-1}.

a Show that the motion is simple harmonic.

b Calculate the amplitude and the period.

12 A component in a machine vibrates with an angular velocity of 120 radians per second and experiences a maximum acceleration of 1000 m s^{-2}. Calculate the maximum displacement and the maximum speed of the component if it is moving in simple harmonic motion.

13 The velocity of a particle moving in a straight line is given by $v = \sqrt{12 + 8x - 4x^2}$, where x is the displacement from a central point.

a Show that the motion is simple harmonic.

b Calculate the amplitude and the period of oscillation.

14 The rise and fall of the tide at a particular inlet is in simple harmonic motion, with the time difference between successive high tides being 10 h. The entrance to the inlet has a depth of 20 m at high tide and 8 m at low tide. If low tide occurs at 10:00 a.m., find the earliest time at which a vessel requiring a minimum water depth of 15 m can pass through the entrance.

CHAPTER SUMMARY
RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

8

- **Implicit differentiation** is the technique of differentiating an equation, involving the dependent and independent variables, using the chain rule for the parts that are functions of the dependent variable.
- The relation between the rates of change of **related variables** with respect to time can be found using the chain rule in the form $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dy}{dt}$.
- A **differential equation** involves one or more derivatives of a function. A **first-order** differential equation involves only the first derivative. The **general solution** of a differential equation is the function.
- The differential equations $\frac{dy}{dx} = ky$ and $\frac{dy}{dx} = g(y)$ have the **general solutions** $y = y_0 e^{kx}$, where y_0 is the value of y at $x = 0$ and $\int \frac{1}{g(y)} dy = x + c$.
- A differential equation with **separable variables** can be rewritten so that the dependent and independent variables are separated. These are often of the form $\frac{dy}{dx} = f(x) \cdot g(y)$.
- A **gradient field** shows what a solution to a differential equation will look like by showing gradients of the functions as short lines or arrows at multiple locations.
- The **logistic model** of growth is given by $\frac{dy}{dt} = ky(M - y)$, where the rate of change of a variable is proportional to both the variable and the difference between a maximum value and the variable.
- The general solution for the logistic model given by $\frac{dy}{dt} = ky(M - y)$ is $y = \frac{MA}{A + e^{-kMt}}$ or $y = \frac{My_0}{y_0 + (M - y_0)e^{-kMt}}$ where $A = \frac{y_0}{M - y_0}$ and y_0 is the initial value.
- **Newton's first law of motion** states that 'Unless acted on by a resultant **force**, a body remains at rest or in uniform motion in a straight line.'
- **Newton's second law of motion** states that 'The acceleration of a body is directly proportional to the resultant force and inversely proportional to the **mass** of the object. The acceleration is in the direction of the resultant force.' Units are chosen so that this law becomes $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ and $\mathbf{a}$ are vectors and m is a scalar.
- The **momentum**, **p**, of a body is defined as the product of its mass and velocity. $\mathbf{p} = m\mathbf{a}$ is a vector. Newton's second law of motion can also be expressed as $\mathbf{F} = \frac{d\mathbf{p}}{dt}$
- **Newton's third law of motion** states that 'For every action there is an equal and opposite reaction.'
- The motion of a body depends on the resultant force acting.
- The equations of motion in a straight line with constant force are as below, where the constant acceleration is a, velocity is v, initial velocity is u, time is t and displacement is s.

  $v = u + at \qquad s = ut + \frac{1}{2}at^2$

  $s = \left(\frac{v + u}{2}\right) \qquad v^2 = u^2 + 2as$

- The acceleration of an object may be expressed in any of the following forms.
  $$a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$
- **Simple harmonic motion** is in a straight line with a force given by $\mathbf{F} = -k\mathbf{x}$, where $\mathbf{x}$ is the displacement of the particle from a fixed point called the **point of equilibrium**.
- The **angular velocity** of a particle undergoing simple harmonic motion is the coefficient of t in the equation $x = a\cos(\omega t + \alpha)$ or $x = a\sin(\omega t + \alpha)$.
- The **amplitude** is the maximum displacement from the equilibrium point.
- The **period** is the time taken for one complete cycle and the **frequency** is the number of cycles in unit time. The **phase** is the angular displacement at $t = 0$.
- The equations for simple harmonic motion with angular velocity ω, amplitude a, period T, frequency f and phase α are:

  Acceleration:

  $$\ddot{x} = -\omega^2 x = -a\omega^2\sin(\omega t + \alpha) \text{ or } -a\omega^2\cos(\omega t + \alpha)$$

  Velocity:

  $$v = \dot{x} = \pm\omega\sqrt{a^2 - x^2} = -a\omega\sin(\omega t + \alpha) \text{ or } a\omega\cos(\omega t + \alpha)$$

  Displacement:

  $$x = a\cos(\omega t + \alpha) \text{ or } x = a\sin(\omega t + \alpha)$$

  Period and frequency: $T = \frac{2\pi}{\omega} = \frac{1}{f}, \omega = 2\pi f$

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CHAPTER REVIEW
RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

8

Multiple choice

1 Example 1 What is $\frac{dy}{dx}$ for $3y^2 - 5x^3 = 4x$?

A $\frac{5x^2}{2y}$ B $-\frac{5x^2}{2y}$ C $15x^2 + 4 - 6y$

D $\frac{15x^2+4}{6y}$ E $\frac{3x^2+4}{2y}$

2 Example 3 The function $y = 4x^2 + 5x$ and $\frac{dx}{dt} = 2$ when $x = 1$. Find $\frac{dy}{dt}$ when $x = 1$.

A -26 B -9 C 9 D 13 E 26

3 Example 6 The general solution to $\frac{dy}{dt} = 30(2t + 1)^2$ is:

A $120(2t + 1) + c$ B $5(2t + 1)^3 + c$ C $10(t^2 + 1)^3 + c$

D $10(t^2 + t)^3 + c$ E $5(t^2 + t)^3 + c$

4 Example 8 A train leaving the station has its acceleration given by $1.8t - 0.21t^2$ for the first 10 seconds of motion. What is the speed and distance moved after 10 seconds?

A 20 m/s and 125 m B 30 m/s and 100 m C 110 m/s and 725 m

D 20 m/s and 380 m E 30 m/s and 200 m

5 Example 9 Solve $\frac{dy}{dx} = 2 - y$

A $y = Ae^{-x} + 2$ B $y = Ae^x - 2$ C $y = 2 - Ae^x$

D $y = 2Ae^{-x}$ E $y = 2e^{-x}$

6 Example 12 Solve $\frac{dy}{dx} = (2x - 5)(y + 3)$

A $y = Ae^{2x-5} - 3$ B $y = Ae^{x^2-5} - 3$ C $y = 3 - Ae^{2x-5}$

D $y = Ae^{2x^2-5} - 3$ E $y = Ae^{2x^2-5x} - 3$

7 Example 17 A force of 40 newtons is applied for 4 seconds to a stationary ball of mass 10 kg. What is the acceleration of the ball 2.5 seconds after the force is applied?

A 1 m/s^2 B 4 m/s^2 C 10 m/s^2 D 16 m/s^2 E 40 m/s^2

8 Example 19 The engine of a car coasting downhill at 60 m/s applied a force to the car, increasing its speed to 84 m/s in 4 seconds. Given that the mass of the car is 900 kg, what force was applied?

A 1500 N B 1800 N C 5400N D $6666.\overline{6}$ N E 86 400 N

Short answer

9 Example 2 An ellipse has the equation $\frac{x^2}{16}+\frac{y^2}{36}=1$. Find the gradient and equation to the tangent at $(2\sqrt{3}, 3)$.

10 Example 4 The piston in a car is at the bottom of a cylinder of diameter 8 cm. At the top of the stroke, the fuel is ignited and the expansion of the gas drives the piston downwards. When the piston is 2 cm from the top, the gas in the cylinder is expanding at a rate of 30 L/s. At what speed is the piston moving down?

11 Example 7 Find the solution to $\frac{dy}{dx}=\frac{1}{x^2-1}$, given that when $x=0$, $y=3$.

12 Example 9 Solve the equation $\frac{dy}{dx}=-0.5y$, given that at $x=0$, $y=8$.

13 Example 14 Solve $\frac{dP}{dV}=P^3V^2$.

14 Example 18 A railway carriage with a mass of 3 tonnes runs into the 'bumper' at the end of the track at 15 m/s and rebounds at 5 m/s and is in contact with the bumper for 0.2 seconds. What force did the bumper exert on the carriage?

15 Example 20 A diver on a high board bounces up and down several times before actually diving into the pool. The high board is 6 m above the pool and the diver takes 2.2 seconds from the time she leaves the board to the time she enters the water.

a How long does it take her to reach her highest point?

b At what speed does she enter the water?

16 Example 21 Two cars in a 100 m drag race have masses of 500 kg and 600 kg and their engines can develop forward forces of 600 000 N and 690 000 N respectively. The lighter car takes off 0.02 s later than the heavier car. Which car crosses the finish line first, and how far ahead of the other car is it at that point?

17 Example 22 Find the speed after 4 seconds of a 12 kg mass that moves from rest under the action of a force given by $F=2t+3$ newtons, where t is in seconds.

18 Example 23 Find an expression for the velocity of a 3 kg particle with an initial velocity of 3 m/s and position of $x=0$ m if the force acting on it is given by $F=2x-1$ newtons.

19 Example 25, 26 A particle starts from rest at $x = 10$ and proceeds with simple harmonic motion about $x = 0$ so that after 2 s it reaches $x = 5$. If displacement is in mm, find:

a an expression for the displacement (x) at any time t

b the speed at $x = 0$

c the amplitude and period of motion

d the maximum speed

e the maximum acceleration.

Application

20 A particle with simple harmonic motion is moving with a velocity of 3.6 m s^{-1} as it passes through its central position. When the particle is 0.2 m from the central position, it has an acceleration of 4.8 m s^{-2}. Calculate:

a the amplitude of the oscillation

b the period of the oscillation.

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21 An express elevator travels 30 floors in 8 s. The distance travelled is 72 m. It accelerates for the first 5 floors and decelerates at the same rate for the last 5 floors, travelling at uniform speed in between. Find that uniform speed.

22 The rate of change in the level of water when it is siphoned between two tanks depends on the difference in the levels and on the size of the hose connecting them. Two rainwater tanks are connected by a hose at 9 a.m., when the difference in their levels is 1.5 m. At 9:05 a.m. the difference in their levels has reduced to 1.2 m. Use a differential equation to find the time when the difference in their levels is less than 1 cm.

23 The maximum population of elephants that can be supported in a reserve is 6000. The population had been reduced by poaching to only 200 elephants before a very strict anti-poaching policy allowed the elephants to recover. The population increased according to the logistic model with $k = 0.000\ 016$. How many years would it take to reach 3000 elephants?

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24 An object of mass 4 kg is projected vertically upwards against resistive forces that are proportional to the square of the speed and equal to 10 N when the speed is 20 m/s. Assuming that $g \approx 10$ m/s^2, find the greatest height reached by the object if its initial velocity is 30 m/s.

Qz Practice quiz